$\dfrac{\text{factors of constant}}{\text{factors of leading coeff.}}$

KeyConcept Rational Zero Theorem

Words If $P(x)$ is a polynomial function with integral coefficients, then every rational zero of $P(x) = 0$ is of the form $\frac{p}{q}$, a rational number in simplest form, where p is a factor of the constant term and q is a factor of the leading coefficient.

Example Let $f(x) = 6x^4 + 22x^3 + 11x^2 - 80x - 40$. If $\frac{4}{3}$ is a zero of $f(x)$, then 4 is a factor of -40, and 3 is a factor of 6.

1. List all possible rational zeros of the following functions:
   a. $f(x) = 4x^5 + x^4 - 2x^3 + 5x^2 + 8x + 16$
   b. $f(x) = x^3 - 2x^2 + 5x + 12$

$\dfrac{\text{factors of } 16}{\text{factors of } 4} = \dfrac{\pm1, \pm16, \pm4, \pm2, \pm8}{\pm1, \pm4, \pm2}$

$\pm1, \pm2, \pm4, \pm8, \pm16, \pm 1/4, \pm 1/2,$

$\dfrac{\text{factors of } 12}{\text{factors of } 1} = \dfrac{\pm1, \pm12, \pm2, \pm6, \pm3, \pm4}{\pm1}$

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

2. Find all the zeros of the following functions:
   a. $h(x) = 9x^4 + 5x^2 - 4$

possible zeros $= \dfrac{\pm1, \pm2, \pm4}{\pm1, \pm3, \pm9} = \pm1, \pm2, \pm4, \pm 1/3, \pm 2/3, \pm 4/3, \pm 1/9, \pm 2/9, \pm 4/9$

Try possibilities
↳ be strategic → dont start w/ fractions
* know $h(1)$ = synthetic division w/ 1 final term

$$\begin{array}{r|rrrrr} 1 & 9 & 0 & 5 & 0 & -4 \\ & & 9 & 9 & 14 & 14 \\ \hline & 9 & 9 & 14 & 14 & 10 \end{array}$$

not a zero

$$\begin{array}{r|rrrrr} \frac{2}{3} & 9 & 0 & 5 & 0 & -4 \\ & & 6 & 4 & 6 & 4 \\ \hline & 9 & 6 & 9 & 6 & 0 \end{array}$$

is a zero

$h(x) = (3x-2)(9x^3 + 6x^2 + 9x + 6)$ * factor by grouping

$= (3x-2)[3x^2(3x+2) + 3(3x+2)]$

$= (3x-2)(3x^2+3)(3x+2)$

$= 3(x^2+1)(3x-2)(3x+2)$

$\boxed{\text{zeros} = \pm i, \pm 2/3}$

b. $k(x) = 2x^4 - 5x^3 + 20x^2 - 45x + 18$

possible zeros $= \dfrac{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18}{\pm 1, \pm 2} = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}$

$$\begin{array}{r|rrrrr} 1 & 2 & -5 & 20 & -45 & 18 \\ & & 2 & -3 & 17 & -28 \\ \hline & 2 & -3 & 17 & -28 & -10 \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 2 & -5 & 20 & -45 & 18 \\ & & 4 & -2 & 36 & -18 \\ \hline & 2 & -1 & 18 & -9 & 0 \checkmark \end{array}$$

$k(x) = (x-2)(2x^3 - x^2 + 18x - 9)$

$= (x-2)[x^2(2x-1) + 9(2x-1)]$ factor by grouping

$= (x-2)(x^2+9)(2x-1)$

$\boxed{\text{zeros: } x = 2, \tfrac{1}{2}, \pm 3i}$

c. $f(x) = 3x^3 - 2x^2 - 8x + 5$

$\dfrac{\pm 1, \pm 5}{\pm 1, \pm 3} = \pm 1, \pm 5, \pm \frac{1}{3}, \pm \frac{5}{3}$

$$\begin{array}{r|rrrr} 1 & 3 & -2 & -8 & 5 \\ & & 3 & 1 & -7 \\ \hline & 3 & 1 & -7 & -2 \end{array}$$

$$\begin{array}{r|rrrr} \frac{5}{3} & 3 & -2 & -8 & 5 \\ & & 5 & 5 & -5 \\ \hline & 3 & 3 & -3 & 0 \checkmark \end{array}$$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -2 & -8 & 5 \\ & & 1 & -\frac{1}{3} & \\ \hline & 3 & -1 & & \end{array}$$

$f(x) = (3x-5)(3x^2 + 3x - 3)$

$= 3(3x-5)(x^2 + x - 1)$ quad form

$x = \dfrac{-1 \pm \sqrt{1^2 + 4(1)(1)}}{2(1)}$

$= \dfrac{-1 \pm \sqrt{5}}{2}$

$\boxed{x = \tfrac{5}{3}, \dfrac{-1 \pm \sqrt{5}}{2}}$

d. $f(x) = 4x^4 + 13x^3 - 8x^2 - 13x - 12$

$\dfrac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 4} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}$

$$\begin{array}{r|rrrrr} 1 & 4 & 13 & -8 & -13 & -12 \\ & & 4 & 17 & 9 & -4 \\ \hline & 4 & 17 & 9 & -4 & -16 \end{array}$$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & 13 & -8 & -13 & -12 \\ & & 2 & & & \\ \hline & 4 & 15 & & & \end{array}$$

$$\begin{array}{r|rrrrr} -2 & 4 & 13 & -8 & -13 & -12 \\ & & -8 & -10 & 36 & -46 \\ \hline & 4 & 5 & -18 & 23 & -58 \end{array}$$

$$\begin{array}{r|rrrrr} -3 & 4 & 13 & -8 & -13 & -12 \\ & & -12 & -3 & 33 & -60 \\ \hline & 4 & 1 & -11 & 20 & \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 4 & 13 & -8 & -13 & -12 \\ & & -4 & -9 & 17 & -4 \\ \hline & 4 & 9 & -17 & 4 & \end{array}$$

e. $f(x) = 8x^3 + 14x^2 + 11x + 3$

$\frac{\pm 1, \pm 3}{\pm 1, \pm 2, \pm 4, \pm 8} = \pm 1, \pm 3, \pm 1/2, \pm 3/2, \pm 1/4, \pm 3/4, \pm 1/8$

$$\begin{array}{r|rrrr} -1 & 8 & 14 & 11 & 3 \\ & & -8 & -6 & -5 \\ \hline & 8 & 6 & 5 & \end{array}$$

$$\begin{array}{r|rrrr} -\tfrac{1}{2} & 8 & 14 & 11 & 3 \\ & & -4 & -5 & -3 \\ \hline & 8 & 10 & 6 & 0 \end{array}$$

$f(x) = (2x+1)(8x^2 + 10x + 6)$

$= 2(2x+1)(4x^2 + 5x + 3)$ prime

$= 2(2x+1)(\quad + 1)(\quad + 3)$

$x = \frac{-5 \pm \sqrt{5^2 - 4(4)(3)}}{2(4)}$

$= \frac{-5 \pm \sqrt{-23}}{8}$

$= \frac{-5}{8} \pm \frac{\sqrt{23}}{8} i$

$\boxed{\text{zeros: } x = -1/2,\ -5/8 \pm \sqrt{23}/8\, i}$

f. $j(x) = 4x^4 - 12x^3 + 25x^2 - 14x - 15$

$\frac{\pm 1, \pm 3, \pm 5, \pm 15}{\pm 1, \pm 2, \pm 4}$

$$\begin{array}{r|rrrrr} -\tfrac{1}{2} & 4 & -12 & 25 & -14 & -15 \\ & & -2 & 7 & -16 & 15 \\ \hline & 4 & -14 & 32 & -30 & 0 \end{array}$$

$j(x) = (2x+1)(4x^3 - 14x^2 + 32x - 30)$

$$\begin{array}{r|rrrr} \tfrac{3}{2} & 4 & -14 & 32 & -30 \\ & & 6 & -12 & 30 \\ \hline & 4 & -8 & 20 & 0 \end{array}$$

$j(x) = (2x+1)(2x-3)(4x^2 - 8x + 20)$

$x = \frac{8 \pm \sqrt{8^2 - 4(4)(20)}}{2(4)}$

$= \frac{8 \pm \sqrt{-256}}{8} = \frac{8 \pm 16i}{8}$

$\boxed{\text{zeros: } -1/2,\ 3/2,\ 1 \pm 2i}$

3. Find the remaining factors of the polynomial $f(x) = x^4 - 2x^3 - 17x^2 + 18x + 72$. $(x-3)$ is a factor of the polynomial.

$$\begin{array}{r|rrrrr} 3 & 1 & -2 & -17 & 18 & 72 \\ & & 3 & 3 & -42 & -72 \\ \hline & 1 & 1 & -14 & -24 & 0 \end{array}$$

$f(x) = (x-3)\underbrace{(x^3 + x^2 - 14x - 24)}$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

$f(x) = (x-3)(x+2)(x^2 - x - 12)$

$\boxed{f(x) = (x-3)(x+2)(x-4)(x+3)}$