Reviewing the Determinate

The determinate of a 2x2 matrix is computed by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinate of a 3x3 matrix is computed by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example: Find the determinate of this matrice.

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 0 & 2 \\ -3 & 6 & 7 \end{vmatrix} = 1 \begin{vmatrix} 0 & 2 \\ 6 & 7 \end{vmatrix} - 3 \begin{vmatrix} 5 & 2 \\ -3 & 7 \end{vmatrix} + 4 \begin{vmatrix} 5 & 0 \\ -3 & 6 \end{vmatrix}$$

$$= 1(0-12) - 3(35+6) + 4(30-0)$$

$$= -12 - 123 + 120$$

$$= -15$$

can also:

$$\left|\begin{matrix} 1 & 3 & 4 \\ 5 & 0 & 2 \\ -3 & 6 & 7 \end{matrix}\right| \begin{matrix} 1 & 3 \\ 5 & 0 \\ -3 & 6 \end{matrix}$$

$$1(0)(7) + 3(2)(-3) + 4(5)(6)$$
$$- 4(0)(-3) - 1(2)(6) - 3(5)(7)$$

$$= 0 - 18 + 120 + 0 - 12 - 105$$

$$= -15$$

https://www.youtube.com/watch?v=eu6i7WJeinw

Properties of the Cross Product: If **a**, **b**, and **c** are vectors and d is a scalar, then

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) = -\mathbf{b} \times \mathbf{a}$
- $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then $\mathbf{a} \times \mathbf{b} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i(a_2b_3 - a_3b_2) - j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)$$

1. Compute the following for the vectors $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, -5, 6 \rangle$.

a. $\mathbf{a} \times \mathbf{b}$

$$\begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & -5 & 6 \end{vmatrix} = i(3(6) - 4(-5)) - j(1(6) - 4(2)) + k(1(-5) - 3(2))$$

$$= 38i + 2j - 11k$$

$$\langle 38, 2, -11 \rangle$$

b. $\mathbf{b} \times \mathbf{a}$

$$\begin{vmatrix} i & j & k \\ 2 & -5 & 6 \\ 1 & 3 & 4 \end{vmatrix} = i(-5(4) - 6(3)) - j(2(4) - 6(1)) + k(2(3) - (-5)(1))$$

$$= -38i - 2j + 11k$$

$$\langle -38, -2, 11 \rangle$$

c. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$

$$a \cdot \langle 38, 2, -11 \rangle$$

$$= \langle 1, 3, 4 \rangle \cdot \langle 38, 2, -11 \rangle$$

$$= 38 + 6 - 44$$

$$= 0$$

* $\vec{a} \times \vec{b}$ perpendicular to $\vec{a}$ & $\vec{b}$
dot product = 0 means $\perp$

2. Find a vector orthogonal to the plane determined by the points $A(1,2,3)$, $B(4,6,8)$ and $C(15,2,-5)$

$\overrightarrow{AB} = \langle 3, 4, 5\rangle$

$\overrightarrow{AC} = \langle 14, 0, -8\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 3 & 4 & 5 \\ 14 & 0 & -8 \end{vmatrix} = i(4(-8) - 0) - j(3(-8) - 5(14)) + k(3(0) - 4(14))$$

$$= -32i + 94j - 56k$$

$$\langle -32, 94, -56\rangle$$

Theorem: The magnitude of $\mathbf{A} \times \mathbf{B}$ is

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta, \text{ where } \theta \text{ is the angle between them}$$
$$= \text{area of the parallelogram spanned by } \mathbf{A} \text{ and } \mathbf{B}.$$

3. The points $P(1, 2, 1)$, $Q(1, 0, 0)$ and $R(0, 3, 1)$ create a triangle. What is the area of the triangle?

Area of $\Delta = \frac{1}{2}$ Area of parallelogram

$\overrightarrow{PQ} = \langle 0, -2, -1\rangle$

$\overrightarrow{PR} = \langle -1, 1, 0\rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 0 & -2 & -1 \\ -1 & 1 & 0 \end{vmatrix} = i(0-1) - j(0-1) + k(0-2)$$
$$= -i + j - 2k$$
$$= \langle -1, 1, -2\rangle$$

$$\frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\left|\langle -1, 1, -2\rangle\right| = \frac{1}{2}\sqrt{(-1)^2 + (1)^2 + (-2)^2} = \frac{\sqrt{6}}{2}$$