Equations of a Line (Point-Direction Form)

The line L through the point $P_0 = (x_0, y_0, z_0)$ in the direction of $\mathbf{v} = \langle a, b, c \rangle$ is described by

Vector Parametrization:

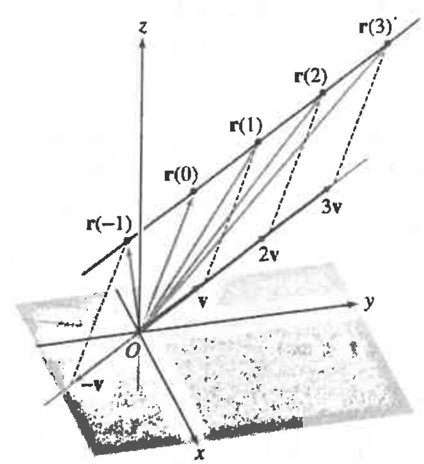
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

where $\mathbf{r}_0 = \overrightarrow{0P_0}$.

Parametric Equations:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

The parameter t takes on values $-\infty < t < \infty$



DF FIGURE 13 The terminal point of $\mathbf{r}(t)$ traces out a line as t varies from $-\infty$ to ∞.

1. Find the vector equation and the parametric equations of a line through the point $(1, 2, 3)$ where the line is parallel to the vector $\mathbf{v} = \langle 2, 5, 10 \rangle$.

$r(t) = \langle 1, 2, 3 \rangle + t\langle 2, 5, 10 \rangle$

$r(t) = \langle 1 + 2t, \ 2 + 5t, \ 3 + 10t \rangle$ *vector eqt

parametric eqt:

$x = 1 + 2t$
$y = 2 + 5t$
$z = 3 + 10t$

2. Find the vector equation of the line through the points $(3, 5, 5)$ and $(2, 1 - 5)$. Also give the parametric equations of this line. Where does the line intersect the xy-plane?

$V = \langle 2 - 3, \ 1 - 5, \ -5 - 5 \rangle$

$= \langle -1, -4, -10 \rangle$

$r(t) = \langle 3, 5, 5 \rangle + t\langle -1, -4, -10 \rangle$

$= \langle 3 - t, \ 5 - 4t, \ 5 - 10t \rangle$

$z = 0$ so $5 - 10t = 0$
$t = 1/2$
$\langle 3 - 1/2, \ 5 - 4(1/2), \ 5 - 10(1/2) \rangle$
$\langle 2.5, \ 3, \ 0 \rangle$

parametric eqts:

$x = 3 - t$
$y = 5 - 4t$
$z = 5 - 10t$

3. Is the point $(7,10,17)$ on the line $\mathbf{r} = \langle 1+3t, 2+4t, 3+7t \rangle$?

$7 = 1 + 3t$

$6 = 3t$

$2 = t$

$r(2) = \langle 1+3(2),\ 2+4(2),\ 3+7(2) \rangle$

$= \langle 7,\ 10,\ 17 \rangle$

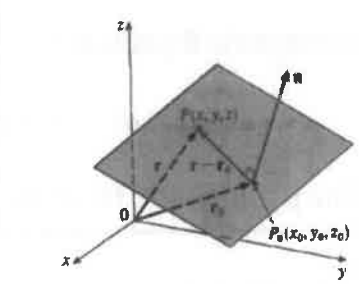
$(7, 10, 17)$ on the line

Plane is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n} = \langle a, b, c \rangle$ that is orthogonal to the plane. The vector $\mathbf{n}$ is called a normal vector.

Vector Form: $\mathbf{n} \cdot \langle x, y, z \rangle = d$

Scalar Form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = d$



4. Find an equation of the plane through the point $(1,2,3)$ and is orthogonal to the vector $\langle 3,4,7 \rangle$

$n = \langle 3, 4, 7 \rangle$ $\qquad \langle x, y, z \rangle = \overrightarrow{OP_0} = \langle 1, 2, 3 \rangle$

$\langle 3, 4, 7 \rangle \cdot \langle 1, 2, 3 \rangle = d$

$3 + 8 + 21 = d$

$32 = d$

$\boxed{3x + 4y + 7z = 32}$

*to get a, b, c recognize $n = \langle a,b,c \rangle = \langle 3,4,7 \rangle$

or compute $n \cdot \langle x, y, z \rangle = 32$

5. Find an equation of the plane through the points $A(1,1,3)$, $B(-1,3,2)$ and $C(1,-1,2)$.

1) Find a normal vector → cross product

$\overrightarrow{AB} = \langle -2, 2, -1 \rangle$

$\overrightarrow{BC} = \langle 2, -4, 0 \rangle$

$\begin{vmatrix} i & j & k \\ -2 & 2 & -1 \\ 2 & -4 & 0 \end{vmatrix} = i(0-4) - j(0+2) + k(8-4)$

$= -4i - 2j + 4k$

$\langle -4, -2, 4 \rangle$

2) Find d

$\langle -4, -2, 4 \rangle \cdot \langle 1, 1, 3 \rangle = d$ ← choose any pt

$-4 - 2 + 12 = d$

$6 = d$

$\boxed{-4x - 2y + 4z = 6}$

6. Find an equation of the plane through the point A $(1, 2, 3)$ and contains the line $x = 2 + 4t,\ y = 1 + 5t,\ z = -1 + 3t$.

$r(t) = \langle 2+4t,\ 1+5t,\ -1+3t \rangle$

1) $\vec{v} = \langle 4, 5, 3 \rangle$ * remember $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

2) Find any other pt
let $t = 0$
$B\ (2, 1, -1)$

3) vector $\overrightarrow{AB} = \langle 1, -1, -4 \rangle$

4) normal vector

$$\vec{v} \times \overrightarrow{AB} = \begin{vmatrix} i & j & k \\ 4 & 5 & 3 \\ 1 & -1 & -4 \end{vmatrix} = i(-20+3) - j(-16-3) + k(-4-5)$$

$$= -17i + 19j - 9k$$

$$= \langle -17, 19, -9 \rangle$$

5) find d $\quad \langle -17, 19, -9 \rangle \langle 1, 2, 3 \rangle = -17 + 38 - 27 = -6$

$\boxed{-17x + 19y - 9z = -6}$

Definition: Two planes are parallel if their normal vectors are parallel.

Definition: Two planes are perpendicular(orthogonal) if their normal vectors are perpendicular.

Definition: The angle between two non-parallel planes is the acute angle between the normal vectors.

7. Determine if there are pairs of planes (listed below) that are parallel, orthogonal, or neither?

$P_1:\ 4x + 2y - 8x = 15 \quad \vec{n}_1 = \langle 4, 2, -8 \rangle$

$P_2:\ 2x + y - 4z = 12 \quad \vec{n}_2 = \langle 2, 1, -4 \rangle$

$P_3:\ 3x + 2y + 2z = 10 \quad \vec{n}_3 = \langle 3, 2, 2 \rangle$

* $\perp$ dot product $= 0$

$\vec{n}_2 \cdot \vec{n}_3 = \langle 2, 1, -4 \rangle \cdot \langle 3, 2, 2 \rangle$
$= 6 + 2 - 8$
$= 0$

$\vec{n}_1 \cdot \vec{n}_3 = \langle 3, 2, 2 \rangle \cdot \langle 4, 2, -8 \rangle$
$= 12 + 4 - 16$
$= 0$

$P_1 \parallel P_2$ b/c $2\vec{n}_2 = \vec{n}_1$

$P_2 \perp P_3$ b/c $\vec{n}_2 \cdot \vec{n}_3 = 0$

$P_1 \perp P_3$ b/c $\vec{n}_1 \cdot \vec{n}_3 = 0$