Theorem Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

Note: If $\mathbf{r}(t)$ is a **position function** of a particle at time t, then the **velocity function** is $\mathbf{r}'(t) = \mathbf{v}(t)$ and the **acceleration function** is $\mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t)$.

Definition: The **unit tangent vector** at t is defined to be $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then.

$$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)) \quad \text{chain rule}$$

Example: Given $\mathbf{r}(t) = \langle 3t,\ e^{2t-4},\ \sin(t\pi)\rangle$.

(a) Find a tangent vector to the curve at $t = 0$.

$$r'(t) = \langle 3,\ 2e^{2t-4},\ \pi\cos(t\pi)\rangle$$

$$r'(0) = \langle 3,\ 2e^{-4},\ \pi\rangle$$

(b) Find $\mathbf{T}(0)$.

$$T(0) = \frac{r'(0)}{|r'(0)|} = \frac{\langle 3,\ 2e^{-4},\ \pi\rangle}{\sqrt{9 + 4e^{-8},\ \pi^2}} = \frac{\langle 3,\ 2e^{-4},\ \pi\rangle}{4.344}$$

$$= \langle 0.69,\ 0.008,\ 0.725\rangle$$

(c) Find a tangent line to the curve at the point $(6, 1, 0)$

$t = 2$

$$L(t) = r(t) + t\, r'(t)$$

* remember eqt

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$\vec{r}_0 = \overrightarrow{OP_0}$ and $\vec{v}$ is vector in direction of line

$$L(2) = r(2) + t\, r'(2)$$

$$= \langle 6, 1, 0\rangle + t\langle 3, 2, \pi\rangle$$

$$= \langle 6+3t,\ 1+2t,\ \pi t\rangle$$

Example: Given $\mathbf{r}(t) = \langle 3t,\ e^{2t-4},\ \sin(t\pi)\rangle$. compute $\int \mathbf{r}(t)\ dt$

$$\int r(t) = \left\langle \int 3t\,dt,\ \int e^{2t-4}\,dt,\ \int \sin(t\pi)\,dt \right\rangle$$

$$= \left\langle \tfrac{3}{2}t^2,\ \tfrac{1}{2}e^{2t-4},\ -\tfrac{1}{\pi}\cos\pi t \right\rangle + \vec{C}$$

$1-\cos t$

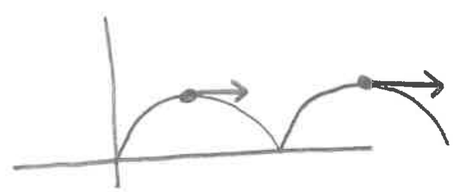
Example: The function $\mathbf{r}(t) = \langle t - \sin(t), \cancel{[illegible]}\rangle$ traces a cycloid. Find the points where:

$r'(t) = \langle 1-\cos t, \sin t\rangle$

a. $\mathbf{r}'(t)$ is horizontal and nonzero.

if $r'(t)$ is horizontal then y - component $= 0$

$\sin t = 0 \qquad t = 0, \pi, \ldots$



$r(0) = \langle 0, 0\rangle \qquad r'(0) = \langle 0, 0\rangle$

$r(\pi) = \langle \pi, 2\rangle \qquad r'(\pi) = \langle 2, 0\rangle$

$r'(t)$ is horizontal & nonzero for $t = (k+1)\pi$ where $\{k \in \mathbb{Z} \mid k \geq 0\}$

$(t = \pi, 3\pi, 5\pi, \ldots)$

b. $\mathbf{r}'(t)$ is the zero vector

$r'(t) = 0$ for $t = 2k\pi$

where $\{k \in \mathbb{Z} \mid k \geq 0\}$

$(t = 0, 2\pi, 4\pi, \ldots)$

Let's prove that if $\mathbf{r}(t)$ has constant length (i.e. $|\mathbf{r}(t)| = c$ (some constant)), then $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$.

$$|r| = \sqrt{a^2+b^2+c^2}$$

$$|r|^2 = \left(\sqrt{a^2+b^2+c^2}\right)^2$$

$$= a^2+b^2+c^2$$

$$= r \cdot r$$

☆ $r = \langle a, b, c \rangle$

$$r \cdot r = \langle a,b,c\rangle \cdot \langle a,b,c\rangle$$

$$= a^2+b^2+c^2$$

$$|r| = \sqrt{a^2+b^2+c^2}$$

we have $r \cdot r = |r|^2 = c^2$ so

$$r \cdot r = c^2$$

$$\frac{d}{dt}(r \cdot r) = \frac{d}{dt}(c^2)$$

$$r' \cdot r + r \cdot r' = 0$$

$$2r' \cdot r = 0$$

$$r' \cdot r = 0$$

know dot product = 0 then perpendicular

Example: Evaluate $\int_{-1}^{3} \langle 8t^2 - t, 6t^3 + t \rangle dt$

$$= \left\langle \tfrac{8}{3}t^3 - \tfrac{1}{2}t^2, \tfrac{3}{2}t^4 + \tfrac{1}{2}t^2 \right\rangle \Big|_{-1}^{3}$$

$$= \left\langle \tfrac{8}{3}(3)^3 - \tfrac{1}{2}(3)^2, \tfrac{3}{2}(3)^4 + \tfrac{1}{2}(3)^2 \right\rangle - \left\langle \tfrac{8}{3}(-1)^3 - \tfrac{1}{2}(-1)^2, \tfrac{3}{2}(-1)^4 + \tfrac{1}{2}(-1)^2 \right\rangle$$

$$= \left\langle 72 - \tfrac{9}{2}, \tfrac{243}{2} + \tfrac{9}{2} \right\rangle - \left\langle -\tfrac{8}{3} - \tfrac{1}{2}, \tfrac{3}{2} + \tfrac{1}{2} \right\rangle$$

$$= \left\langle \tfrac{212}{3}, 124 \right\rangle$$

Example: Find the location and velocity at $t = 4$ of a particle whose path satisfies

$$\frac{d\mathbf{r}}{dt} = \langle 2t^{-1/2}, 6, 8t \rangle \qquad \mathbf{r}(1) = \langle 4, 9, 2 \rangle$$

velocity $\rightarrow$ $\left.\frac{dr}{dt}\right|_{t=4} = \boxed{\langle 1, 6, 32 \rangle}$

$$r(t) = \int_1^t \frac{dr}{du}\, dt = \int_1^t \langle 2u^{-1/2}, 6, 8u \rangle$$

$$= \left.\langle 4u^{1/2}, 6u, 4u^2 \rangle\right|_1^t$$

$$= \langle 4t^{1/2}, 6t, 4t^2 \rangle - \langle 4, 6, 4 \rangle$$

$$= \langle -4 + 4t^{1/2}, -6 + 6t, -4 + 4t^2 \rangle$$

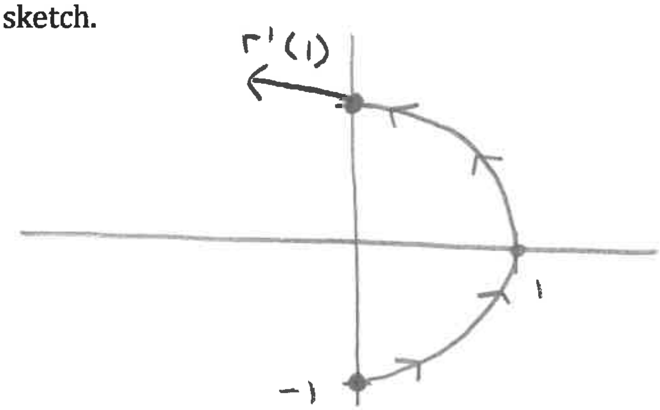
$\boxed{r(4) = \langle 4, 18, 60 \rangle}$

Example: Sketch the curve parametrized by $\mathbf{r}(t) = \langle 1 - t^2, t \rangle$ for $-1 \le t \le 1$. Compute the tangent vector at $t = 1$ and add it to the sketch.

$r(t) = \langle 1 - t^2, t \rangle$

$x = 1 - t^2 \qquad y = t$

$x = 1 - y^2$



$r'(t) = \langle -2t, 1 \rangle$

$r'(1) = \langle -2, 1 \rangle$