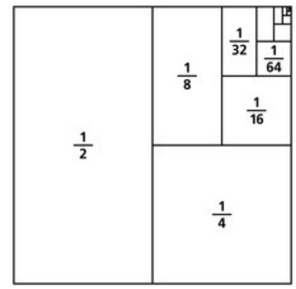
Overview:

One infinite process that had puzzled mathematicians for centuries was the summing of infinite series. Sometimes an infinite series of terms added up to a number:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1$$



But sometimes the infinite sum was infinite as in:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots=\infty$$

And sometimes the infinite sum was difficult to evaluate:

$$1-1+1-1+1-1+1-1+\ldots$$

(Is it 0? Is it 1? Is it neither?)

Roughly Speaking Definitions:

Sequence: list of numbers written in a definite order
Ex. $1,\ 2,\ 3,\ 4,\ldots$

vs.

Series: the operation of adding infinitely many quantities one after another.
Ex. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

Sequence Converges: individual terms approach the same number.

ex) $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ldots$ sequences converges to 0

Series Converges: sum of the terms approach a number:

Infinite Series:
An infinite series is an expression of the form

$$a_1+a_2+a_3+\ldots+a_n+\ldots,\ \text{or}\ \sum_{k=1}^{\infty} a_k$$

The numbers $a_1,\ a_2,\ \ldots$ are the **terms** of the series; a_n is the n**th term**.

The **partial sums** of the series form a sequence

$$s_1=a_1$$
$$s_2=a_1+a_2$$
$$\vdots$$
$$s_n=\sum_{k=1}^{n} a_k$$
$$\vdots$$

of real numbers, each defined as a finite sum. If the sequence of partial sums has a limit S as $n\to\infty$ we say the series **converges** to the sum S, and we write

$$a_1+a_2+a_3+\ldots+a_n+\ldots=\sum_{k=1}^{\infty} a_k=S$$

Otherwise, we say the series **diverges.**

1. Does the series $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + ...$ converge?

2. Does the series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ... + \frac{3}{10^n} + ...$ converge?

3. Compute the limit of the partial sums, $2 - 1 + 1 - 1 + 1 - 1 + ...$, to determine whether the series converges or diverges.

The **geometric series**

$$a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ... = \sum_{n=1}^{\infty} ar^{n-1}$$

Converges to the sum $\frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$

The interval ______________________ is the **interval of convergence** for a geometric series.

3. Tell whether each series converges or diverges. If it converges, give its sum.

a. $\sum_{n=1}^{\infty} 3(\frac{1}{2})^{n-1}$

c. $\sum_{k=0}^{\infty} (\frac{3}{5})^k$

b. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + (-\frac{1}{2})^{n-1} + ...$

d. $\frac{\pi}{2} + \frac{\pi^2}{2} + \frac{\pi^3}{2} + ...$

We can have a series that represents a function and converges to the function!. Let's take a look at:

$$1 + x + x^2 + x^3 +.. + x^n$$

A. Write the function this series represents

B. Open Desmos and graph the function you wrote in #1 above. Then graph some of the partial sums.

C. Does the series converge to the function you wrote in #1? If so, over what interval does the series converge?

Throughout this chapter we are going to look at series that represent functions. Writing functions as series helps us find antiderivatives, model functions, find function values, approximate functions with much more precision than Euler's method, etc.

ex) Can you find $\int \cos(x^3)dx$?

4. Find the interval of convergence and the function of x represented by the geometric series:

a. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

b. $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

Derivatives and Integrals of Series:

Given a series you can take the derivative and the integral of each individual term as well as the general term.

If $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$, then $f'(x) =$

A $\frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} + \cdots + \frac{x^{(2n+1)}}{(2n+1)n!} + \cdots$

B $x + \frac{3x^3}{2!} + \frac{5x^5}{3!} + \frac{7x^7}{4!} + \cdots + \frac{(2n-1)x^{(2n-1)}}{n!} + \cdots$

C $2 + 2x^2 + x^4 + \frac{x^6}{3} + \cdots + \frac{2x^{2(n-1)}}{(n-1)!} + \cdots$

D $2x + 2x^3 + x^5 + \frac{x^7}{3} + \cdots + \frac{2nx^{(2n-1)}}{n!} + \cdots$

Extra AP Question:

The Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\frac{x^2}{1-x^2}$?

A $1 + x^2 + x^4 + x^6 + x^8 + \cdots$

B $x^2 + x^3 + x^4 + x^5 + \cdots$

C $x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots$

D $x^2 + x^4 + x^6 + x^8 + \cdots$

E $x^2 - x^4 + x^6 - x^8 + \cdots$