BC Calculus
10.1 Power Series

Overview:

One infinite process that had puzzled mathematicians for centuries was the summing of infinite series. Sometimes an infinite series of terms added up to a number:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$$

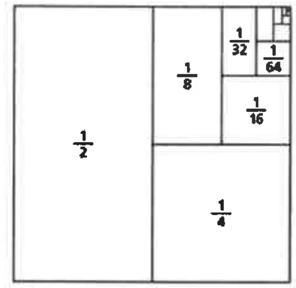
But sometimes the infinite sum was infinite as in:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \infty$$

And sometimes the infinite sum was difficult to evaluate:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

(Is it 0? Is it 1? Is it neither?)



Roughly Speaking Definitions:

Sequence: list of numbers written in a definite order
Ex. $\{1, 2, 3, 4\}$

vs.

Series: the operation of adding infinitely many quantities one after another.
i.e. the sum of the terms of an infinite sequence is an infinite series

Sequence Converges: individual terms approach the same number.
ex) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ sequences converges to 0

sequence $a_n = \frac{1}{n}$ converges to 0
$1, 1/2, 1/3, 1/4, \ldots$ $\lim_{n\to\infty} \frac{1}{n} = 0$

Series Converges: sum of the terms approach a number:

series $1/n$ diverges
$1 + 1/2 + 1/3 + 1/4 + \ldots$ doesn't have finite sum

Infinite Series:
An infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots, \text{ or } \sum_{k=1}^{\infty} a_k$$

The numbers $a_1, a_2, \ldots$ are the **terms** of the series; a_n is the n **th term**.

The **partial sums** of the series form a sequence

↳ list of terms

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

partial sums approach a value as $n \to \infty$ to converge

of real numbers, each defined as a finite sum. If the sequence of partial sums has a limit S as $n \to \infty$ we say the series **converges** to the sum S, and we write

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k = S$$

Otherwise, we say the series **diverges.**

1. Does the series $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$ converge? tempted to group:

$(1-1) + (1-1) + (1-1) + (1-1) + \ldots$ but requires ∞ # of pairings and cannot be justified

* sequence of partial sums must converge

$s_1 = 1$

$s_2 = 1 - 1 = 0$

$s_3 = 1 - 1 + 1 = 1$

$s_4 = 1 - 1 + 1 - 1 = 0$

sequence of partial sums $= 1, 0, 1, 0, \ldots$ sequence diverges so series has no sum $\boxed{\text{diverges}}$

2. Does the series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots + \frac{3}{10^n} + \ldots$ converge?

$s_1 = 3/10$

$s_2 = 3/10 + 3/100 = 33/100$

$s_3 = 3/10 + 3/100 + 3/100 = 333/1000$

$\frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \ldots$

sequence has a limit $0.\overline{3}$ or $1/3$

$\boxed{\text{series converges to } 1/3}$

3. Compute the limit of the partial sums, $2 - 1 + 1 - 1 + 1 - 1 + \ldots$, to determine whether the series converges or diverges.

$s_1 = 2$

$s_2 = 2 - 1 = 1$

$s_3 = 2 - 1 + 1 = 2$

$s_4 = 2 - 1 + 1 - 1 = 1$

$2, 1, 2, 1, \ldots$

sequence diverges so $\boxed{\text{series diverges}}$

The geometric series

$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$

Converges to the sum $\frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$

* only series we can find the sum for

The interval $-1 < r < 1$ is the **interval of convergence** for a geometric series.

3. Tell whether each series converges or diverges. If it converges, give its sum.

a. $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$

$a_1 = 3$

$r = 1/2$ so converges

$\frac{3}{1 - 1/2} = 6$

b. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \left(-\frac{1}{2}\right)^{n-1} + \ldots$

$r = -1/2$ so converges

$\frac{1}{1-(-1/2)} = \frac{2}{3}$

c. $\sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k$

$a_1 = 1$

$r = 3/5$ so converges

$\frac{1}{1 - 3/5} = \frac{5}{2}$

d. $\frac{\pi}{2} + \frac{\pi^2}{2} + \frac{\pi^3}{2} + \ldots$

$r = \pi$ so diverges

We can have a series that represents a function and converges to the function!. Let's take a look at:

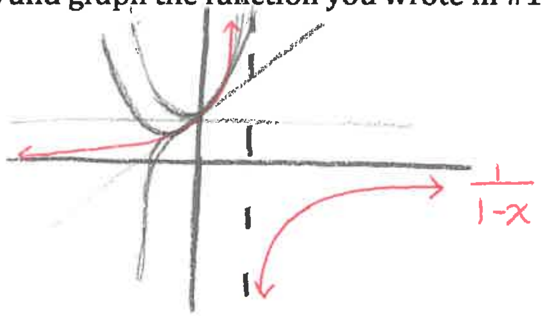
$$1 + x + x^2 + x^3 + .. + x^n$$

* this series called a power series

A. Write the function this series represents

$$f(x) = \frac{a_1}{1-r} = \frac{1}{1-x}$$

B. Open Desmos and graph the function you wrote in #1 above. Then graph some of the partial sums.



$S_1 = 1$

$S_2 = 1+x$

$S_3 = 1+x+x^2$

$S_4 = 1+x+x^2+x^3$

C. Does the series converge to the function you wrote in #1? If so, over what interval does the series converge?

yes but only when $x<1$ and seems to be around $x>-1$

* notice $r = x$ in geo series

$-1 < r < 1$

$-1 < x < 1$

Throughout this chapter we are going to look at series that represent functions. Writing functions as series helps us find antiderivatives, model functions, find function values, approximate functions with much more precision than Euler's method, etc.

ex) Can you find $\int \cos(x^3)dx$? no but we can by writing $\cos(x^3)$ as a series!

4. Find the interval of convergence and the function of x represented by the geometric series:

a. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

$r = -(x+1)$

write a few terms to confirm r

$= (-1)^0(x+1)^0 + (-1)^1(x+1)^1 + (-1)^2(x+1)^2 + \dots$

$= 1 - (x+1) + (x+1)^2 - \dots$

$a_1 = 1 \quad r = -(x+1)$

$$f(x) = \frac{a_1}{1-r} = \frac{1}{1-[-(x+1)]} = \frac{1}{x+2}$$

$-1 < -(x+1) < 1$

$1 > x+1 > -1$

$0 > x > -2$

$-2 < x < 0$ interval of convergence

b. $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n = 3 + 3\left(\frac{x-1}{2}\right) + 3\left(\frac{x-1}{2}\right)^2 + \dots$

$r = \frac{x-1}{2} \quad a_1 = 3$

$$f(x) = \frac{3}{1-\frac{x-1}{2}} = \frac{3}{\frac{2-x+1}{2}} = \frac{6}{-x+3}$$

$-1 < \frac{x-1}{2} < 1$

$-2 < x-1 < 2$

$-1 < x < 3$

Derivatives and Integrals of Series:

Given a series you can take the derivative and the integral of each individual term as well as the general term.

If $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$, then $f'(x) =$

A $\frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} + \cdots + \frac{x^{(2n+1)}}{(2n+1)n!} + \cdots$

B $x + \frac{3x^3}{2!} + \frac{5x^5}{3!} + \frac{7x^7}{4!} + \cdots + \frac{(2n-1)x^{(2n-1)}}{n!} + \cdots$

C $2 + 2x^2 + x^4 + \frac{x^6}{3} + \cdots + \frac{2x^{2(n-1)}}{(n-1)!} + \cdots$

D $2x + 2x^3 + x^5 + \frac{x^7}{3} + \cdots + \frac{2nx^{(2n-1)}}{n!} + \cdots$

$f(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots + \frac{x^{2n}}{n!}$

$f'(x) = 2x + \frac{4x^3}{2} + \frac{6x^5}{6} + \frac{8x^7}{24} + \dots + \frac{2nx^{2n-1}}{n!}$

$= 2x + 2x^3 + x^5 + \frac{x^7}{3} + \frac{2nx^{2n-1}}{n!}$

Extra AP Question:

~~The Maclaurin series~~ for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Which of the following is a power series expansion for $\frac{x^2}{1-x^2}$?

can ignore

A $1 + x^2 + x^4 + x^6 + x^8 + \cdots$

B $x^2 + x^3 + x^4 + x^5 + \cdots$

C $x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots$

D $x^2 + x^4 + x^6 + x^8 + \cdots$

E $x^2 - x^4 + x^6 - x^8 + \cdots$

$\frac{x^2}{1-x^2} = \frac{a_1}{1-r}$ * recognize form

$a_1 = x^2$

$r = x^2$

$x^2 + x^4 + x^6 + x^8 + \dots$