Recall from Khan Video
$f(a)$ gives same point at $a$
$f'(a)$ gives same slope at $a$
$f''(a)$ gives same concavity at $a$



**Taylor Series Generated by $f$ at $x = a$**
Let $f$ be a function with derivatives of all orders throughout some open interval containing $a$. Then the Taylor series generated by $f$ at $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + .. + \frac{f^{(n)}(a)}{n!}(x-a)^n + ... = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-a)^k.$$

The partial sum

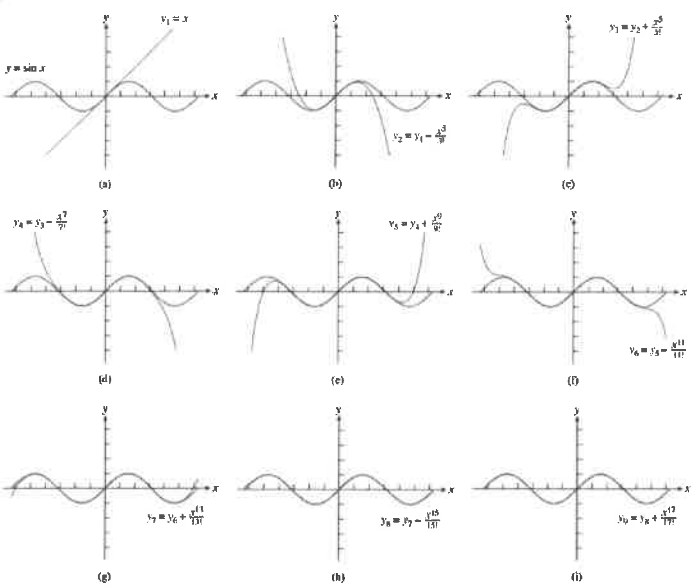
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the Taylor polynomial of order $n$ for $f$ at $x = a$.

Purpose of Taylor Series:
- Approximate solutions to very difficult/impossible differential equations
- Determining various infinite sums (coming soon!)
- Energy functions in physics
- Many engineering applications

* Mclaurin series is expansion of Taylor series centered at 0

1. Construct the seventh-order Taylor polynomial and the Taylor series for $\sin x$ at $x = 0$.



$\sin(0) = 0$
$\sin'(0) = \cos 0 = 1$
$\sin''(0) = -\sin 0 = 0$
$\sin'''(0) = -\cos(0) = -1$
$\sin^{(4)}(0) = \sin 0 = 0$
$\sin^{(5)}(0) = \cos 0 = 1$
$\vdots$

pattern $0, 1, 0, -1$ keeps repeating

$$P_7(x) = 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 - \frac{1}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4 + \frac{1}{5!}(x-0)^5 + \frac{0}{6!}(x-0)^6 - \frac{1}{7!}(x-0)^7$$

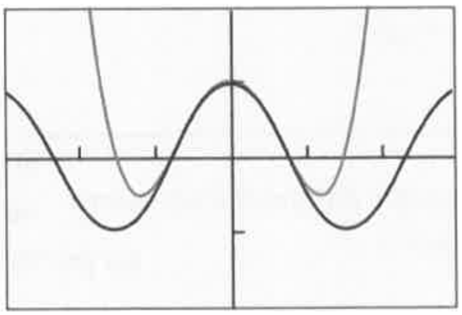
$$\boxed{P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}$$

*Even the billionth-order Taylor polynomial begins to peel away from $\sin x$ as we move away from 0. Can approximate the sine of any number to whatever accuracy we want if we work out enough terms of this series!

2. Find the fourth-order Taylor [polynomial that approximates $y = \cos 2x$ near $x = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$$



[−3, 3] by [−2, 2]

**Figure 9.5** The graphs of $y = 1 - 2x^2 + (2/3)x^4$ and $y = \cos 2x$ near $x = 0$. (Example 3)

$$P_4(x) = 1 - \frac{4x^2}{2} + \frac{16x^4}{24}$$

$$\boxed{P_4(x) = 1 - 2x^2 + \frac{2x^4}{3}}$$

approximates so well close to 0!

3. Find the Taylor series generated by $f(x) = e^x$ at $x = 2$

$f(2) = e^2$

$f'(2) = e^2$

$f''(2) = e^2$

$f'''(2) = e^2$

$f^{(4)}(2) = e^2$

$\vdots$

$$e^x = e^2 + e^2(x-2) + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{3!} + \cdots + \frac{e^2(x-2)^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{e^2(x-2)^k}{k!}$$

* check graph to see convergence around $x = 2$

4. Find the third-order Taylor polynomial for $f(x) = 2x^3 - 3x^2 + 4x - 5$ at

a. $x = 0$

* same as polynomial

$f(0) = -5$

$f'(0) = 6(0)^2 - 6(0) + 4 = 4$

$f''(0) = 12(0)^1 - 6 = -6$

$f'''(0) = 12$

$f''''(0) = 0$

$$f(x) = -5 + 4x - \frac{6x^2}{2!} + \frac{12x^3}{3!}$$

$$\boxed{= -5 + 4x - 3x^2 + 2x^3}$$

b. $x = 1$

$f(1) = 2(1)^3 - 3(1)^2 + 4(1) - 5 = -2$

$f'(1) = 6(1)^2 - 6(1) + 4 = 4$

$f''(1) = 12(1) - 6 = 6$

$f'''(1) = 12$

$$P_3(x) = -2 + 4(x-1) + \frac{6(x-1)^2}{2!} + \frac{12(x-1)^3}{3!}$$

$$\boxed{P_3(x) = -2 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3}$$

* function same → check by multiplying out

On the intersections of their intervals of convergence, Taylor series can be added, subtracted, and multiplied by constants and powers of $x$, and the results are Taylor series.

5. Find a Maclaurin series to represent the function $f(x) = \frac{e^{2x}-1}{3}$. What is the interval of convergence?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots + \frac{(2x)^n}{n!} + \dots$$

$$e^{2x} - 1 = 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots + \frac{(2x)^n}{n!} + \dots$$

$$\frac{e^{2x}-1}{3} = \frac{1}{3}\left(2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots + \frac{(2x)^n}{n!} + \dots\right)$$

$$= \frac{2}{3}x + \frac{(2x)^2}{3\cdot 2!} + \frac{(2x)^3}{3\cdot 3!} + \frac{(2x)^4}{3\cdot 4!} + \dots + \frac{(2x)^n}{3\cdot n!} + \dots$$

$e^x$ converges for $\mathbb{R}$ so $\frac{e^{2x}-1}{3}$ converges for $\mathbb{R}$

6. Use the formula in the definition to construct the fifth-order Taylor polynomial and the Taylor series for the function, $f(x) = e^{1-x}$, at $x = 0$

$f(0) = e^1 = e$

$f'(0) = -e^{1-x} = -e$

$f''(0) = e^{1-x} = e$

$f'''(0) = -e^{1-x} = -e$

$f^{(4)}(0) = e$

OR $P_{4,e^x}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$P_{4,e^{-x}}(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$

$* \; e^1 e^{-x} = e^{1-x}$

$P_{4,e^{1-x}}(x) = e - ex + \frac{ex^2}{2!} - \frac{ex^3}{3!} + \frac{ex^4}{4!}$

$$P_4(x) = e - e(x-0) + \frac{e(x-0)^2}{2!} - \frac{e(x-0)^3}{3!} + \frac{e(x-0)^4}{4!} - \frac{e(x-0)^5}{5!}$$

$$= e - ex + \frac{ex^2}{2!} - \frac{ex^3}{3!} + \frac{ex^4}{4!} - \dots + \frac{(-1)^n ex^n}{n!} + \dots$$

**Maclaurin Series (Taylor Series at $x = 0$)**

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad (|x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (\text{all real } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad (\text{all real } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad (\text{all real } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad (-1 < x \leq 1)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (|x| \leq 1)$$

For AP test must know: $\sin x$, $\cos x$, $e^x$ as it provides the foundation for constructing the Maclaurin series for other functions.

7. The $n^{th}$ derivative of $g$ at $x = 0$ is given by $g^{(n)}(0) = \frac{\sqrt{n+7}}{n^3}$ for $n \geq 1$. What is the coefficient for the term containing $x^2$ in Maclaurin series of $g$.

$$f(a) + f'(a)(x-a) + \underbrace{\frac{f''(a)(x-a)^2}{2!}} + \frac{f'''(x-a)^3}{3!} + \ldots$$

$$g''(0) = \frac{\sqrt{2+7}}{2^3} = \frac{3}{8} \qquad \frac{\frac{3}{8}}{2!} = \boxed{\frac{3}{16}}$$

8. Use the table of MAclaurin series. Construct the first three nonzero terms and the general term of the Maclaurin series generated by the function and give the interval of convergence.

a. $f(x) = 7xe^x$

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

$$7x + 7x^2 + \frac{7x^3}{2!} + \ldots + \frac{7x \cdot x^n}{n!}$$

$$\boxed{7x + 7x^2 + \frac{7x^3}{2!} + \ldots + \frac{7x^{n+1}}{n!}}$$

convergent for $\mathbb{R}$

b. $f(x) = x^2 \cos x$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$$

$$x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} + \ldots + \frac{(-1)^n x^2 x^{2n}}{(2n)!} \ldots$$

$$\boxed{x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} + \ldots + \frac{(-1)^n x^{2n+2}}{(2n)!} + \ldots}$$

converges for $\mathbb{R}$