Alternating Series Error Bound

Suppose the terms of a series have the following three properties:

1. The terms alternate in sign;
2. The terms decrease in absolute value;
3. The terms approach 0 as a limit.

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Then the truncation error after n terms is less than the absolute value of the $(n+1)$ st term.

Lagrange Error Bound

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Let's say we will approximate $f(x)$ with a 3rd order polynomial.

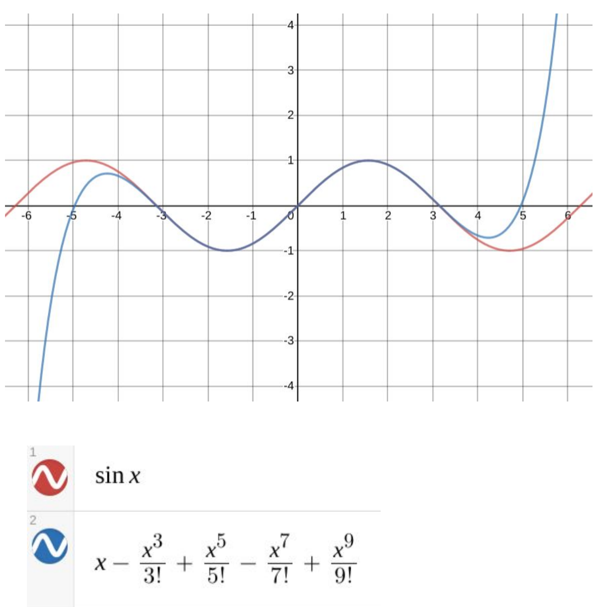
$$f(x) \approx P_3(x)$$

$$f(x) = P_3(x) + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \ldots.$$

$$f(x) = P_3(x) + Remainder$$

$$f(x) = P_3(x) + R_n(x)$$

$$f(x) - P_3(x) = R_n(x)$$



$$R_n(x) \leq |f(x) - P_n(x)|$$

Bounding the Remainder: $R_n(x) \leq \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

*(n+1)st derivative max that derivative could be between where the taylor series is centered, a, and where the polynomial is approximating the function, x. Lagrange error not exact, only provides bound on error.

If $R_n(x) \to 0$ as $n \to \infty$ for all x in the interval, we say that the Taylor series generated by f at $x = a$ **converges to** f on the interval.

1. Suppose we use the first two terms of the Maclaurin series for $f(x) = \ln(1+x)$ to approximate $f(0.1)$. Find a bound on the error by using (a) the alternating series error bound, and (b) the Lagrange error formula.

2. Suppose the second-order Taylor polynomial for $f(x)$ about $x = 2$ is used as an approximation for $f(x)$, and suppose that $|f^{(3)}(x)| < 2.3$ for all x in the interval $[2,\ 2.5]$. Show that $|f(x) - P_2(x)| < 0.05$ for all x in the interval $[2,\ 2.5]$.

3. Use Lagrange error to prove that $|f(0.2) - P(0.2)| < \frac{1}{1000}$ for the fourth degree Taylor polynomial centered at 0 to approximate $f(0.2)$ for the function $f(x) = e^x$.

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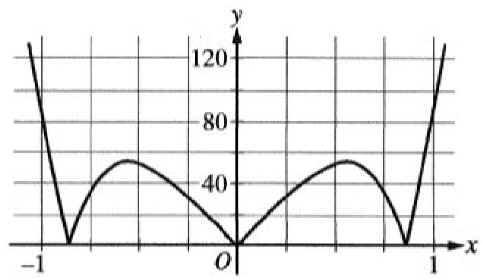
Question 6



Graph of $y = \left|f^{(5)}(x)\right|$

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.