Alternating Series Error Bound

Suppose the terms of a series have the following three properties:

1. The terms alternate in sign;
2. The terms decrease in absolute value;
3. The terms approach 0 as a limit.

error $< |a_{n+1}|$

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Then the truncation error after n terms is less than the absolute value of the $(n+1)$ st term.

Lagrange Error Bound

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Let's say we will approximate $f(x)$ with a 3rd order polynomial.

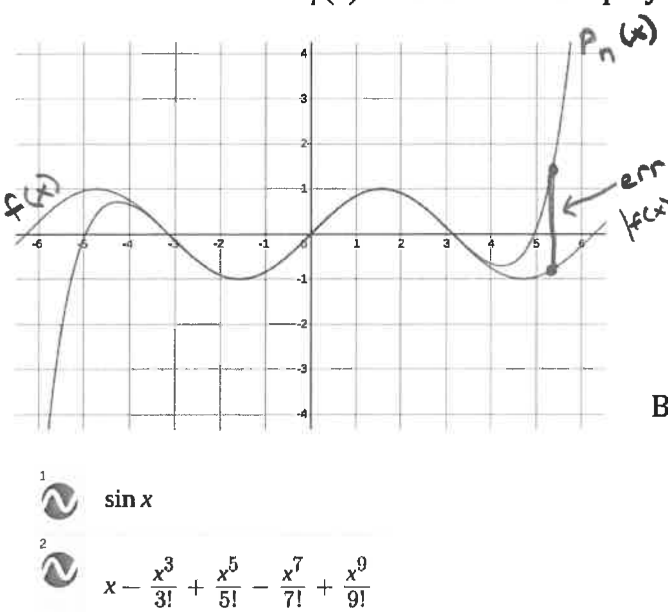
$$f(x) \approx P_3(x)$$

$$f(x) = P_3(x) + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \ldots.$$

$$f(x) = P_3(x) + Remainder$$

$$f(x) = P_3(x) + R_n(x)$$

$$f(x) - P_3(x) = R_n(x)$$



$$R_n(x) \le |f(x) - P_n(x)|$$

really just next term in series after $P_n(x)$

Bounding the Remainder: $R_n(x) \le \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

*(n+1)st derivative max that derivative could be between where the taylor series is centered, a, and where the polynomial is approximating the function, x. Lagrange error not exact, only provides bound on error.

If $R_n(x) \to 0$ as $n \to \infty$ for all x in the interval, we say that the Taylor series generated by f at $x = a$ **converges to** f on the interval.

* $f(x) = P_n(x) +$ Remainder

if too large then can't sum series

centered @ 0

1. Suppose we use the first two terms of the Maclaurin series for $f(x) = \ln(1+x)$ to approximate $f(0.1)$. Find a bound on the error by using (a) the alternating series error bound, and (b) the Lagrange error formula.

$$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

$$P_2(x) = x - \frac{x^2}{2}$$

2 b/c second order coincidence
also 2 terms

$$P_2(0.1) \approx f(0.1)$$

meets 3 criteria for alternating series error bound

$$P_2(0.1) = 0.1 - \frac{0.1^2}{2}$$

$$= 0.095$$

a) $$\text{error} < |a_{n+1}|$$

$$= |a_3|$$

$$= \left|\frac{x^3}{3}\right| = \left|\frac{0.1^3}{3}\right| = \boxed{3.33 \times 10^{-4}}$$

b) $$\text{error} \le \left|\frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!}\right|$$

$$= \left|\frac{f^{(3)}(c)\,(x-0)^3}{3!}\right|$$

$$= \left|\frac{2}{3!}(0.1-0)^3\right|$$

$$\star\ \frac{2}{3!} = \frac{2}{6} = \frac{1}{3}\ \star$$

$$= \frac{0.1^3}{3}$$

$$= \boxed{3.33 \times 10^{-4}}$$

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$f'''(c) =$ largest der can be between 0 and 0.1

$$= \frac{2}{(1+0)^3} = 2$$

2. Suppose the second-order Taylor polynomial for $f(x)$ about $x = 2$ is used as an approximation for $f(x)$, and suppose that $|f^{(3)}(x)| < 2.3$ for all x in the interval $[2,\ 2.5]$. Show that $|f(x) - P_2(x)| < 0.05$ for all x in the interval $[2,\ 2.5]$.

$$|f(x) - P_2(x)| = \text{error} \le \left| \frac{f^{(3)}(c)}{3!} (x-a)^3 \right|$$

$$= \left| \frac{f^{(3)}(c)}{6} (x-2)^3 \right|$$

where approx. since an interval want error to be a bound so use value in interval that will make error max

$$< \left| \frac{2.3}{6} (2.5-2)^3 \right|$$

$$= \left| \frac{2.3}{6} (1/2)^3 \right|$$

$$= \left| \frac{2.3}{48} \right| < 0.05$$

3. Use Lagrange error to prove that $|f(0.2) - P(0.2)| < \frac{1}{1000}$ for the fourth degree Taylor polynomial centered at 0 to approximate $f(0.2)$ for the function $f(x) = e^x$.

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\text{error} = |f(0.2) - P_4(0.2)| \le \left| \frac{f^{(5)}(c)}{5!} (x-0)^5 \right|$$

$$= \left| \frac{f^{(5)}(c)}{5!} (0.2-0)^5 \right|$$

*what is the max value of the fifth derivative of $f(x) = e^x$ on $0 \le x \le 0.2$ would be $e^{0.2}$ but don't know that value → choose a bound you can calculate $e^1 \approx 2.7 \approx 3$

$$< \left| \frac{3}{5!} (0.2)^5 \right|$$

$$= \left| \frac{3}{120} \cdot \left(\frac{1}{5}\right)^5 \right|$$

$$= \frac{1}{40 \cdot 5^5} = \frac{1}{40 \cdot 3125} < \frac{1}{1000}$$

*even a gross overestimate of max derivative yields an error bound $< \frac{1}{1000}$

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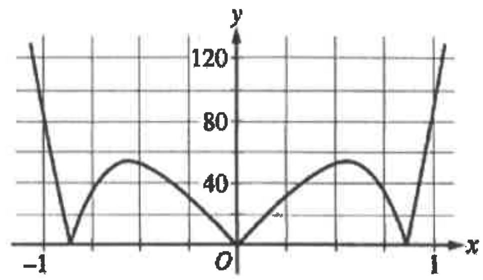
Question 6



Graph of $y = \left|f^{(5)}(x)\right|$

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$

b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$f(x) = \sin(x^2) + \cos x = 1 - \frac{x^2}{2} + x^2 + \frac{x^4}{4!} - \frac{x^6}{6!} - \frac{x^6}{3!} + \ldots$

$= 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{720} - \frac{x^6}{6} + \ldots$

$= 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{720} + \ldots$

c) since $\frac{f^{(6)}(0)x^6}{6!} = \frac{-121}{6!}x^6$

$f^{(6)}(0) = -121$

d) $\left| P_4(1/4) - f(1/4) \right| = \text{error} \leq \left| \frac{f^{(5)}(c)}{5!} (x-0)^5 \right|$

$$= \left| \frac{f^{(5)}(c)\,(1/4)^5}{5!} \right|$$

max $f^{(5)}(c)$ in interval $0 \leq x \leq 1/4$

is 40

* remember finding error bound

makes happens @ $x = 1/4$ and

looks like it might be 30 but

you don't know the actual value

$$< \left| \frac{40}{5!} \left(\frac{1}{4}\right)^5 \right|$$

$$= \frac{40}{120 \cdot 4^5}$$

$$= \frac{1}{3 \cdot 1024}$$

$$= \frac{1}{3072} < \frac{1}{3000}$$