Convergence for a :

Geometric Series:
converges for $|r|<1$
diverges for $|r|\geq 1$

Maclaurin series for $\sin x$, $\cos x$, e^x :
converges for all real x

Nth-Term Test for Divergence

Recall from 10.1 infinite series:

The nth-term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. *The converse of this **is not** true. If $\lim_{n\to\infty} a_n = 0$ the series **could** converge.

1. Determine if the series diverges:

a. $\sum_{n=0}^{\infty} 2^n$

$\lim_{n\to\infty} 2^n = \infty \neq 0$

diverges by nth term test

b. $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$

$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2} \neq 0$

diverges by nth term test

c. $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Inconclusive, the series might converge and might diverge

Direct Comparison Test

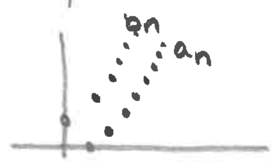
The direct comparison test is a tool that we can use to determine convergence for complicated, **positive** series by comparing them with simpler series.

Direct Comparison Test
Let $0 < a_n < b_n$ for all n

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

a_n (being smaller) cannot exceed b_n

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.



b_n (being larger) cannot sum to a value less than a_n

2. Determine convergence or divergence.

a. $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

$\frac{1}{2+3^n} < \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$

$= \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

converges by the Geometric test since $r = \frac{1}{3}$ and $\left|\frac{1}{3}\right| < 1$. Therefore $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ also converges by direct comparison.

b. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$

$\frac{x^{2n}}{(n!)^2} \le \frac{x^{2n}}{n!}$

$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$

* Taylor series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

so $\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$ is the taylor series for e^{x^2} which will converge for $\mathbb{R}$. Therefore $\sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}$ also converges by direct comparison.

c. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!+2}$

$\frac{x^{2n}}{n!+2} < \frac{x^{2n}}{n!}$

$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

converges for $\mathbb{R}$ x since it is the Taylor series for e^{x^2}. Therefore $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!+2}$ also converges by the direct comparison test

Definitions of Absolute and Conditional Convergence

A series **converges absolutely** (is absolutely convergent) if $\sum |a_n|$ converges.

3. Does the series converge absolutely or diverge?

a. $\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n^2+n}{2}}}{3^n}$

$= -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \cdots$

$\left| \frac{(-1)^{\frac{n^2+n}{2}}}{3^n} \right| \leq \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges by

the geometric test as $r = \frac{1}{3}$

so $\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n^2+n}{2}}}{3^n}$ converges

absolutely.

b. $\sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$

$\left| \frac{(\sin x)^n}{n!} \right| \leq \frac{1}{n!}$

$\sum_{n=0}^{\infty} \frac{1}{n!}$

which converges to e

* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

so $\sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$ converges

absolutely.

Ratio Test

Ratio test

Let $\sum a_n$ be a series with nonzero terms

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$
2. $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$
3. The ratio test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

The ratio test is particularly useful for series that converge rapidly (i.e. factorials or exponentials).

4. Determine Convergence or Divergence:

a. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$a_{n+1} = \frac{2^{n+1}}{(n+1)!}$ $\quad a_n = \frac{2^n}{n!}$

$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right|$

$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}$

$\lim_{n\to\infty} \frac{2\cdot 2^n}{(n+1)n!}\cdot\frac{n!}{2^n}$

$\lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

The series converges by the ratio test

b. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2 2^{n+2}}{3^{n+1}}}{\frac{n^2 2^{n+1}}{3^n}}\right|$

$\lim_{n\to\infty} \frac{(n+1)^2 2^{n+2}}{3^{n+1}}\cdot\frac{3^n}{n^2 2^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)^2\cdot 2}{3\cdot n^2}$

$\lim_{n\to\infty} \frac{2n^2+4n+2}{3n^2} = \frac{2}{3} < 1$

The series converges by the ratio test

Ratio Test

Ratio test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$
2. $\sum a_n$ diverges if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$
3. The ratio test is inconclusive if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

The ratio test is particularly useful for series that converge rapidly (i.e. factorials or exponentials).

4. Determine Convergence or Divergence:

a. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

b. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

c. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$\lim_{n\to\infty} \frac{(n+1)^n (n+1)}{(n+1)\, n!} \cdot \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

the series diverges by the ratio test

if didn't recognize e:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \qquad \infty^{\infty}$$

$$y = \left(1+\frac{1}{n}\right)^n$$

$$\ln y = n \ln(1 + 1/n)$$

$$\lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} \qquad \frac{0}{0}$$

L'hosp.

$$\lim_{n\to\infty} \frac{\frac{1}{1+1/n}\left(-\frac{1}{n^2}\right)}{(-1/n^2)}$$

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

$$\ln y = 1$$

$$y = e^1$$

$$y = e$$

d. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}} \right|$$

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n+1}{n+2}$$

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2}$$

$$= \sqrt{1} \cdot 1 = 1$$

The ratio test is inconclusive

$\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges at $x = a$, which assures us of one point where the series must converge.

The Convergence Theorem for Power Series

There are three possibilities for $\sum_{n=0}^{\infty} c_n(x-a)^n$ with respect to convergence:

1. There is a positive number R such that the series diverges for $|x-a| > R$ but converges for $|x-a| < R$.
   *The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$
2. The series converges for every x ($R = \infty$)
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$)

*We find the radius of convergence by using the Ratio Test

R is radius of convergence

Radius of Convergence for a power series tells you how far away from the center you can be and still find a good approximation

5. Find the Radius and Interval of Convergence for:

a. $\sum_{n=0}^{\infty} n!x^n$ * $n!(x-0)^n$ $a=0$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(n+1)n!\,x^n x}{n!\,x^n}\right|$$

$$\lim_{n\to\infty}|x(n+1)| = |x\cdot\infty| = \infty$$

$\infty > 1$ always

center $= 0$ $R = 0$

The series only converges at the center $x = 0$

b. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $a = 0$

$$\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+1}x^2}{(2n+3)(2n+2)(2n+1)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = \left|\frac{x^2}{\infty}\right| = 0 < 1$$ always

center $= 0$ $R = \infty$

The series converges for every x, $(-\infty, \infty)$

c. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n\,2^n}$ $a = 2$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(x-2)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)^n (x-2)}{(n+1)\,2^n \cdot 2} \cdot \frac{n\,2^n}{(x-2)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)\,n}{(n+1)\,2} \right| = \frac{\infty}{\infty}$$

L'hosp

$$\lim_{n\to\infty} \left| \frac{x-2}{2} \right| = \left| \frac{x-2}{2} \right|$$

$$\left| \frac{x-2}{2} \right| < 1$$

*ratio test <1

$$-1 < \frac{x-2}{2} < 1$$

$$-2 < x-2 < 2$$

$$0 < x < 4$$

radius of convergence $R = 2$ center $= 2$

$$\boxed{\text{interval of convergence } 0 < x < 4}$$

d. $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)\,4^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+2}}{(n+2)\,4^{n+2}} \cdot \frac{(n+1)\,4^{n+1}}{(x-3)^{n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}(x-3)}{(n+2)\,4^{n+1}\,4} \cdot \frac{(n+1)\,4^{n+1}}{(x-3)^{n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-3)(n+1)}{(n+2)\,4} \right| \quad \frac{\infty}{\infty}$$

L'hosp

$$\lim_{n\to\infty} \left| \frac{x-3}{4} \right| = \left| \frac{x-3}{4} \right|$$

$$\left| \frac{x-3}{4} \right| < 1$$

$$-1 < \frac{x-3}{4} < 1$$

$$-4 < x-3 < 4$$

$$-1 < x < 7$$

radius of convergence $R = 4$ center $= 3$

$$\boxed{\text{interval of convergence } -1 < x < 7}$$

* dont know how to check endpoints yet