Integral Test

Use the integral Test to determine whether an infinite series converges or diverges

The Integral Test
If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

1. Use the integral test to determine convergence or divergence of each series.

a. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx$$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\lim_{b\to\infty} \int_1^b \frac{1}{u}\,du$$

$$= \frac{1}{2}\lim_{b\to\infty} \ln|u| \Big|_1^b$$

$$= \frac{1}{2}\lim_{b\to\infty} \ln|x^2+1| \Big|_1^b$$

$$= \frac{1}{2}\lim_{b\to\infty} \left[\ln(b^2+1) - \ln(2)\right]$$

$$= \frac{1}{2}\lim_{b\to\infty} \ln\left(\frac{b^2+1}{2}\right)$$

$$= \infty$$

The series diverges by the integral test b/c $\int_1^{\infty} \frac{x}{x^2+1}\,dx$ diverges

b. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \tan^{-1}x \Big|_1^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1}b - \tan^{-1}(1)\right]$$

$$= \lim_{b\to\infty} \tan^{-1}b - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

The series converges by the integral test b/c $\int_1^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{4}$

c. $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx$$

$u = \ln x$
$du = 1/x\,dx$

$$= \lim_{b\to\infty} \int_2^b u^{-1}\,du$$

$$= \lim_{b\to\infty} \ln|\ln x| \Big|_2^b$$

$$= \lim_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 2)\right]$$

$$= \infty - \ln(\ln 2)$$

The series diverges by the integral test b/c $\int_1^{\infty} \frac{1}{x\ln x}\,dx$ diverges

P-Series Test

Definition of a p-Series
A p-series is a type of series that follows the following pattern:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots.$$

where p is a positive constant. For $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots.$ is called the **harmonic series.**

Convergence of p-Series
The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$

1. Converges if $p > 1$
2. Diverges if $0 < p \leq 1$

2. Determine if the series are convergent or divergent:

a. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$p = 2 > 1$

The series converges by the p-series test

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$p = \frac{1}{2} < 1$

The series diverges by the p-series test

c. $\sum_{n=1}^{\infty} \frac{1}{n}$

$p = 1 \leq 1$

The series diverges by the p-series test

why:

Take $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$$\underbrace{1}_{1} + \underbrace{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}}_{\text{\# greater than 1}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{12}}_{\text{\# greater than 1}} + \underbrace{\frac{1}{13} + \ldots + \frac{1}{34}}_{\text{\# greater than 1}} + \ldots$$

series diverges (very slowly)

Limit Comparison Test

Some series closely resemble others but you are unable to apply the Direct Comparison Test. If this is the case, there is a second comparison test called the Limit Comparison Test.

$\sum_{n=0}^{\infty} \frac{1}{2+\sqrt{n}}$ is a good example where direct comparison will not work but limit comparison will.

> **Limit Comparison Test**
> Suppose that $a_n > 0,\ b_n > 0$ and
> $$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$
> where L is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.
> *For L to be finite and positive it means L cannot be zero or ∞

3. Choosing what to compare:

a. $\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$ $\quad \frac{1}{n^2}$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$ $\quad \frac{1}{\sqrt{n}}$

c. $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{3n-2}}$ $\quad \frac{n^2}{\sqrt{n}} = n^{3/2}$

4. Determine the convergence or divergence of the following series:

a. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ compare to $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}/n^2}{\sqrt{n}/n^2+1} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2} \cdot \frac{n^2+1}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{n^2+1}{n^2} \quad \frac{\infty}{\infty}$$

L'hosp.

$$= \lim_{n\to\infty} \frac{2n}{2n}$$

L'hosp

$$= \lim_{n\to\infty} \frac{2}{2} = 1$$

* L finite and positive

> $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-series test b/c $p = 3/2 > 1$
> so $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges by limit comparison test

b. $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$ compare to $\frac{n2^n}{n^3} = \frac{2^n}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{n2^n}{4n^3+1}}{\frac{n2^n}{n^3}} = \lim_{n\to\infty} \frac{n^3}{4n^3+1} = \frac{1}{4}$$ * L finite and positive

$\boxed{\begin{array}{l} \sum_{n=1}^{\infty} \frac{2^n}{n^2} \text{ diverges by the } n\text{th term test} \\ \text{So, } \sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1} \text{ diverges by the limit comparison test} \end{array}}$

c. $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-2}}$ compare to $\frac{1}{\sqrt[3]{n^2}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^2}}}{\frac{1}{\sqrt[3]{n^2-2}}} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2-2}}{\sqrt[3]{n^2}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n^2-2}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2} - \frac{2}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt[3]{1 - \frac{2}{n^2}}$$

$$= \sqrt[3]{1-0}$$

$$= 1$$

$\boxed{\begin{array}{l} \sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2}} \text{ diverges by p-series } p = \frac{2}{3} \le 1 \\ \text{So, } \sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-2}} \text{ diverges by limit comparison test} \end{array}}$

Alternating Series Test

Most of the tests that we've used so far have dealt with only positive terms (geometric test withstanding).

Recall:
A series whose terms switch between positive and negative is called an **alternating series**

Alternating Series Test
Let $a_n > 0$. The alternating series:

$$\sum_{n=1}^{\infty}(-1)^n a_n \text{ and } \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

will converge if the following two conditions are met

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ for all n

☆ notice not take $(-1)^n$ and $(-1)^{n+1}$ into account only part w/o sign

*If the test fails the first condition, then the series diverges by the nth term test!

5. Use the alternating series test to determine convergence or divergence

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

1) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

2) $\frac{1}{n+1} \le \frac{1}{n}$ ✓

The series converges by alternate series test

b. $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}} = \sum_{n=1}^{\infty}\frac{n}{(-1)^{n-1}(2)^{n-1}} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2^n\,2^{-1}} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{2^n}$

1) $\lim_{n\to\infty}\frac{2n}{2^n} = 0$ ✓

2) $\frac{2(n+1)}{2^{n+1}} \le \frac{2n}{2^n}$ ✓

The series converges by the alternating series test

c. $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(\pi x)$

alternating part

$\cos\pi$ $\cos 2\pi$ $\cos 3\pi$

-1 $+1$ -1

1) $\lim_{n\to\infty} \frac{1}{n^2} = 0$ ✓

2) $\frac{1}{(n+1)^2} \le \frac{1}{n^2}$ ✓

converges by the alternating series test

d. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$

1) $\lim_{n\to\infty} \frac{n+1}{n} = 1$

series diverges by the alternating series test

Conditional Convergence

A series converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

A conditionally convergent series converges only on the condition that it alternates (classic example: harmonic series) whereas absolutely convergent series will converge whether it alternates or not.

6. Does the series converge absolutely, converge conditionally, or diverge?

a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ $p = \frac{2}{3} \le 1$ series diverges by the p-series test

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ converges conditionally

7. Find the interval of convergence for:

a. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ ★ Ratio Test ★

$\lim_{n\to\infty} \left| \frac{x^{n+2}}{n+2} \cdot \frac{n+1}{x^{n+1}} \right|$

$\lim_{n\to\infty} \left| \frac{x^n x^2}{n+2} \cdot \frac{n+1}{x^n x} \right|$

$\lim_{n\to\infty} \left| \frac{x(n+1)}{n+2} \right| = |x| < 1$

$-1 < x < 1$

Let $x = -1$

$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1}$

$\sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1}$

$\sum_{n=0}^{\infty} \frac{-1}{n+1}$

$= -1 + -\frac{1}{2} + -\frac{1}{3} + -\frac{1}{4} \ldots$

diverges by the p-series test b/c $p = 1 \leq 1$

Let $x = 1$

$\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{n+1}}{n+1}$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

Alternating Series Test

1) $\lim_{n\to\infty} \frac{1}{n+1} = 0$ ✓

2) $\frac{1}{n+2} \leq \frac{1}{n+1}$ ✓

converges by the alt. series Test

$\boxed{\text{interval of convergence } (-1, 1]}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}(x-2)^n}{n}$

$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x-2)n}{n+1} \right|$

$= |x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

Let $x = 1$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}(1-2)^n}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}(-1)^n}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^{2n+2}}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$

diverges by p-series test b/c $p = 1 \leq 1$

Let $x = 3$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}(3-2)^n}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}(1)^n}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n}$

Alternating Series Test

1) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

2) $\frac{1}{n+1} \leq \frac{1}{n}$ ✓

converges by the alternating series test

$\boxed{\text{interval of convergence } (1, 3]}$

Finding the Right Test

Test	Series	Condition(s) of Convergence	Condition(s) of divergence	Comment
nth-Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence!
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \frac{a}{1-r}$; sum must start at zero
P-Series	$\sum_{n=0}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Integral Test	$\sum_{n=1}^{\infty} a_n$	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)dx$ diverges	f is continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\sum_{n=1}^{\infty} b_n$ diverges	L must be positive and finite (not zero, not infinity)
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^n a_n$	1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} < a_n$		Remainder: $\lvert R_n\rvert \leq a_{n+1}$
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	Inconclusive if: $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$