Integral Test

Use the integral Test to determine whether an infinite series converges or diverges

The Integral Test
If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

1. Use the integral test to determine convergence or divergence of each series.

   a. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

   b. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

   c. $\sum_{n=2}^{\infty} \frac{1}{n\ \ln(n)}$

P-Series Test

Definition of a p-Series A p-series is a type of series that follows the following pattern: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots.$ where p is a positive constant. For $p = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots.$ is called the <u>**harmonic series.**</u>
Convergence of p-Series The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ 1. Converges if $p > 1$ 2. Diverges if $0 < p \leq 1$

2. Determine if the series are convergent or divergent:
    a. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
    b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
    c. $\sum_{n=1}^{\infty} \frac{1}{n}$

Limit Comparison Test

Some series closely resemble others but you are unable to apply the Direct Comparison Test. If this is the case, there is a second comparison test called the Limit Comparison Test.

$\sum_{n=0}^{\infty} \frac{1}{2+\sqrt{n}}$ is a good example where direct comparison will not work but limit comparison will.

> **Limit Comparison Test**
> Suppose that $a_n > 0,\ b_n > 0$ and
> $$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$
> where L is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.
> *For L to be finite and positive it means L cannot be __________ or _____________

3. Choosing what to compare:

a. $\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$

c. $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{3n-2}}$

4. Determine the convergence or divergence of the following series:

a. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

b. $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$

c. $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-2}}$

Alternating Series Test

Most of the tests that we've used so far have dealt with only positive terms (geometric test withstanding).

Recall:
A series whose terms switch between positive and negative is called an **alternating series**

Alternating Series Test
Let $a_n > 0$. The alternating series:

$$\sum_{n=1}^{\infty}(-1)^n a_n \text{ and } \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

will converge if the following two conditions are met

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ for all n

*If the test fails the first condition, then the series diverges by the nth term test!

5. Use the alternating series test to determine convergence or divergence
   a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$
   b. $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$
   c. $\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(\pi x)$

d. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$

Conditional Convergence

A series converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

A conditionally convergent series converges only on the condition that it alternates (classic example: harmonic series) whereas absolutely convergent series will converge whether it alternates or not.

6. Does the series converge absolutely, converge conditionally, or diverge?

   a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$

7. Find the interval of convergence for:

   a. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

   b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}(x-2)^n}{n}$

c. $\sum_{n=0}^{\infty} x^n$

Finding the Right Test

Test	Series	Condition(s) of	Condition(s) of	Comment

		Convergence	divergence	
nth-Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence!
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum = $\frac{a}{1-r}$; sum must start at zero
P-Series	$\sum_{n=0}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Integral Test	$\sum_{n=1}^{\infty} a_n$	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)dx$ diverges	f is continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\sum_{n=1}^{\infty} b_n$ diverges	L must be positive and finite (not zero, not infinity)
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^n a_n$	1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} < a_n$		Remainder: $\|R_n\| \leq a_{n+1}$
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$	Inconclusive if: $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$