11.1 Parametric Functions
BC Calculus

Parametric Function:

Instead of defining the points (x, y) on a planar curve by relating y directly to x, we can define both coordinates as functions of a parameter t.

1. Make a table of values and sketch the curve, indicating the direction of your graph. Then, eliminate the parameter.

a. $x = \sqrt{t-1},\ y = t+2$

t	1	2	5	10
x	0	1	2	3
y	3	4	7	12

$x = \sqrt{t-1}$

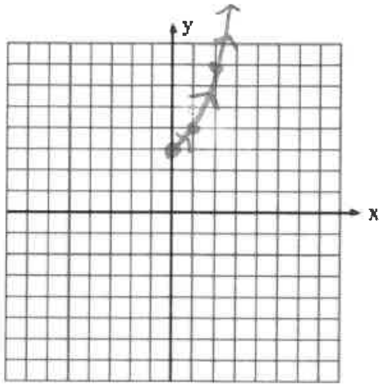
$x^2 = t-1$

$x^2+1 = t$

$y = t+2$

$y = (x^2+1)+2$

$\boxed{y = x^2+3}$



b. $x = t^2 - 3$ and $y = 2t$, $-2 \le t \le 3$

t	-2	-1	0	1	2
x	1	-2	-3	-2	1
y	-4	-2	0	2	4

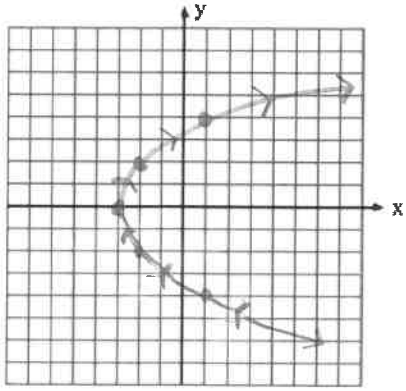
$y = 2t$

$\frac{1}{2}y = t$

$x = t^2 - 3$

$x = (\frac{1}{2}y)^2 - 3$

$\boxed{x = \frac{1}{4}y^2 - 3}$



c. $x = 3 + 2\cos t,\ y = -1 + 3\sin t$

t	0	π/2	π	3π/2	2π
x	5	3	1	3	5
y	-1	2	-1	-4	-1

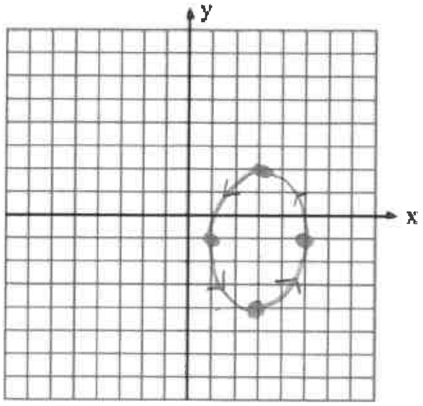
$\cos^2 t + \sin^2 t = 1$

$x = 3 + 2\cos t$ $\quad$ $y = -1 + 3\sin t$

$\frac{x-3}{2} = \cos t$ $\quad$ $\frac{y+1}{3} = \sin t$

$$\left(\frac{x-3}{2}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

$$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{9} = 1$$



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Slope and Concavity

For parametric function the slope of the curve is stil $\frac{dy}{dx}$, and the concavity still depends on $\frac{d^2y}{dx^2}$, so all that is needed is a way of differentiating with respect to x when everything is given in terms of t.

Parametric Differentiation Formulas

If x and y are both differentiable functions of t and if $\frac{dy}{dx} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

If $y' = \frac{dy}{dx}$ is also differentiable functions of t, then

$$\frac{d^2y}{dx^2} = \underbrace{\frac{d}{dx}\left[\frac{dy}{dx}\right]}_{\text{implicit diff.}} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\underbrace{\left(\frac{dt}{dx}\right)}_{\text{chain rule}} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \text{ — invert}$$

2. Given the parametric functions $x = 2\sqrt{t}$ and $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$y = 3t^2 - 2t$ $\quad$ $x = 2\sqrt{t}$

$\frac{dy}{dt} = 6t - 2$ $\quad$ $\frac{dx}{dt} = t^{-1/2}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t-2}{t^{-1/2}}$$

$$\boxed{= 6t^{3/2} - 2t^{1/2}}$$

Formula

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(6t^{3/2} - 2t^{1/2}\right)}{dx/dt} = \frac{9t^{1/2} - t^{-1/2}}{t^{-1/2}} = \boxed{9t - 1}$$

No Formula

$$\frac{d}{dx}\left(6t^{3/2} - 2t^{1/2}\right) = 9t^{1/2}\frac{dt}{dx} - t^{-1/2}\frac{dt}{dx} = \left(9t^{1/2} - t^{-1/2}\right)\frac{dt}{dx} = \frac{\left(9t^{1/2} - t^{-1/2}\right)}{dx/dt} = \frac{9t^{1/2} - t^{-1/2}}{t^{-1/2}}$$

$$\boxed{= 9t - 1}$$

3. Given the parametric equations $x = 4\cos t$ and $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$

$x = 4\cos t$ $\quad$ $y = 3\sin t$

$\frac{dx}{dt} = -4\sin t$ $\quad$ $\frac{dy}{dt} = 3\cos t$

$$\frac{dy}{dx} = \frac{3\cos t}{-4\sin t} = -\frac{3}{4}\cot(t)$$

$$\left.\frac{dy}{dx}\right|_{t=3\pi/4} = -\frac{3}{4}\cot\left(3\pi/4\right) = -3/4(-1) = 3/4$$

$$x\left(3\pi/4\right) = 4\cos 3\pi/4 = 4\left(-\sqrt{2}/2\right) = -2\sqrt{2}$$

$$y\left(3\pi/4\right) = 3\sin\left(3\pi/4\right) = 3\left(\sqrt{2}/2\right) = \frac{3\sqrt{2}}{2}$$

$$\boxed{y - \frac{3\sqrt{2}}{2} = \frac{3}{4}\left(x + 2\sqrt{2}\right)}$$

4. Find the equation of the line tangent to the curve given by $x = 2 - 3\cos\theta$ and $y = 3 + 2\sin\theta$ at the point $(-1, 3)$.

$-1 = 2 - 3\cos\theta \qquad 3 = 3 + 2\sin\theta$

$1 = \cos\theta \qquad 0 = \sin\theta$

$\theta = 0$

$y = 3 + 2\sin\theta \qquad x = 2 - 3\cos\theta$

$\frac{dy}{d\theta} = 2\cos\theta \qquad \frac{dx}{d\theta} = 3\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta}$$

$$= \frac{2}{3}\cot\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \frac{2}{3}\cot(0)$$

$$= \frac{2}{3}(\text{und})$$

= undefined ⇒ vertical tangent line

$\boxed{x = -1}$

5. Find all points of of horizontal and vertical tangency given the parametric equations $x = t^2 + t$, $y = t^2 - 3t + 5$

where slope = 0

$\frac{dx}{dt} = 2t + 1 \qquad \frac{dy}{dt} = 2t - 3$

$$\frac{dy}{dx} = \frac{2t - 3}{2t + 1}$$

vertical tangent : $\frac{\#}{0} \rightarrow \frac{dx}{dt} = 0$

horizontal tangent : $\frac{0}{\#} \rightarrow \frac{dy}{dt} = 0$

vertical tangent → $\frac{dx}{dt} = 0$

$2t + 1 = 0$

$t = -1/2$

$x(-1/2) = (-1/2)^2 - 1/2$

$= -1/4$

$y(-1/2) = (-1/2)^2 - 3(-1/2) + 5$

$= 1/4 + 3/2 + 5$

$= \frac{27}{4}$

$\boxed{(-1/4,\ 27/4)}$

horizontal tangent → $\frac{dy}{dt} = 0$

$2t - 3 = 0$

$t = 3/2$

$x(3/2) = (3/2)^2 + 3/2$

$= 15/4$

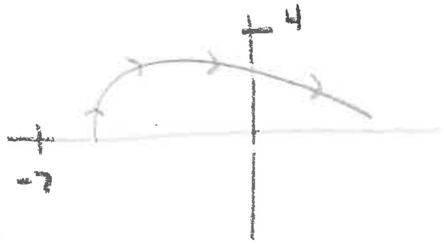
$y(3/2) = (3/2)^2 - 3(3/2) + 5$

$= 11/4$

$\boxed{(15/4,\ 11/4)}$

6. Consider the curve defined parametrically by $x = t^2 - 5$ and $y = 2\sin t$ for $0 \le t \le \pi$.
   a. Sketch a graph of the curve in the viewing window $[-7, 7]$ by $[-4, 4]$. Indicate the direction in which it is traced.
   b. Find the highest point on the curve.
   c. Find all points of inflection on the curve.

Graph
Menu → #3 → #4



★ know direction b/c when t gets larger so does x

b) highest pt on curve = maximum of y

$y = 2\sin t$

$\frac{dy}{dt} = 2\cos t$

$0 = 2\cos t$

$0 = \cos t \qquad t = \pi/2$

$\begin{array}{c|c} + & - \\ \hline \multicolumn{2}{c}{\pi/2} \end{array}$ dy/dt

max @ $t = \pi/2$

$x(\pi/2) = (\pi/2)^2 - 5 = -2.533$

$y(\pi/2) = 2\sin(\pi/2) = 2$

$\boxed{(-2.533,\ 2)}$

c) $\frac{d^2y}{dx^2} = ?$

$\frac{dy}{dt} = 2\cos t \qquad x = t^2 - 5$

$\frac{dx}{dt} = 2t$

$\frac{dy}{dx} = \frac{2\cos t}{2t} = \frac{\cos t}{t}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\cos t / t)}{dx/dt} = \frac{\frac{t(-\sin t) - (\cos t)(1)}{t^2}}{2t}$$

$$\to \frac{-t\sin t - \cos t}{t^2} \cdot \frac{1}{2t} = \frac{-t\sin t - \cos t}{2t^3}$$

$\frac{d^2y}{dx^2} = 0$ and changes sign

$t = 2.798386$

$x(2.798386) = 2.831 \qquad y(2.798386) = 0.673$

$\boxed{(2.831,\ 0.673)}$

Arc Length of a Parametric Curve

Let L by the length of a parametric curve that is traverse exactly once as t increases from t_1 to t_2.

If $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous function of t, then

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

6. A particle moves along the smooth curve given by $x = t^2 + 1$ and $y = 4t^3 - 1$. How far did the particle travel between $t = 0$ and $t = 5$?

$\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = 12t^2$

$$L = \int_0^5 \sqrt{(2t)^2 + (12t^2)^2}\, dt$$

$= 500.815$