Vector Functions and Space Curves

Let $\mathbf{r}$ be a **vector function** whose domain is a set of real numbers and result is a three-dimensional vector. Let

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where $f(t)$, $g(t)$, and $h(t)$ are real valued functions and are called the component functions of $\mathbf{r}$.

The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions:

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

A vector function $\mathbf{r}$ is continuous if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous.

Example: Given $\mathbf{r}(t) = \left\langle t\sqrt{t+5},\ t^2+2,\ \dfrac{e^t - 1}{t} \right\rangle$

a) Find the domain of $\mathbf{r}(t)$.

$t\sqrt{t+5} \to t \geq -5$

$t^2 + 2 \to \mathbb{R}$

$\dfrac{e^t - 1}{t} \to t \neq 0$

domain

$[-5, 0) \cup (0, \infty)$

b) Find all t where $\mathbf{r}(t)$ is continuous.

continuous for

$[-5, 0) \cup (0, \infty)$

c) Compute $\lim_{t\to 0} \mathbf{r}(t)$.

$$\lim_{t\to 0} r(t) = \left\langle \lim_{t\to 0} t\sqrt{t+5},\ \lim_{t\to 0} t^2+2,\ \lim_{t\to 0} \frac{e^t - 1}{t} \right\rangle$$

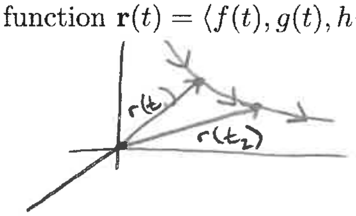
$$= \left\langle 0,\ 2,\ \lim_{t\to 0} \frac{e^t}{1} \right\rangle$$ * L'hospitals

$$= \langle 0, 2, 1 \rangle$$

Definition: Suppose that $f(t)$, $g(t)$, and $h(t)$ are real valued functions on an interval I, then the set C defined as :

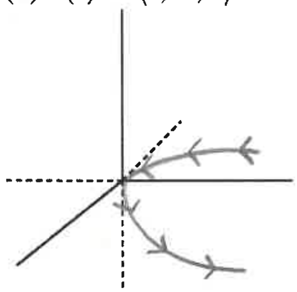
$$C = \{(x, y, z) | x = f(t), y = g(t), z = h(t)\}$$

where t is a parameter and t varies in some interval, I, is called a **space curve**. The space curve C can be traversed by the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.



Example: Describe the curve defined by the vector function. Indicate the direction of motion.

(a) $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$



$x = t$

$y = t^2$

$z = 0$

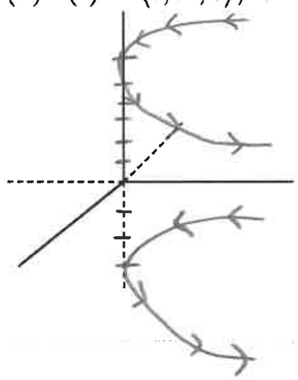
can make a table and choose t values

x	y	z

or eliminate parameter

$y = x^2$

(b) $\mathbf{r}(t) = \langle t, t^2, c \rangle$, where c is a constant.



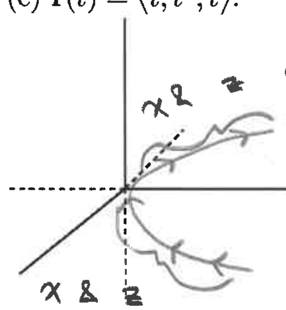
same as (a)

just z any value

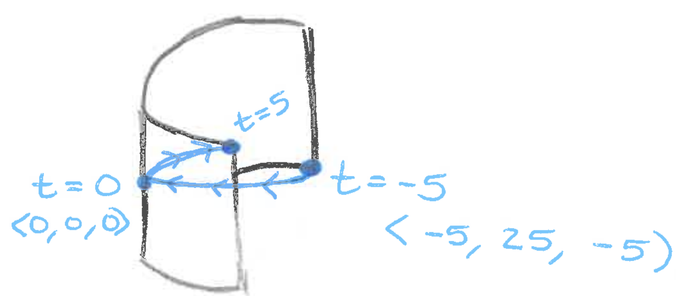
if $z = 6$

if $z = -3$

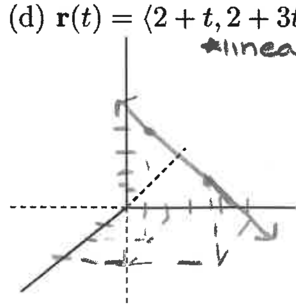
(c) $\mathbf{r}(t) = \langle t, t^2, t \rangle$.



$y = x^2$ but $z = t$

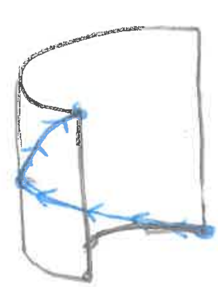


(d) $\mathbf{r}(t) = \langle 2+t, 2+3t, 4-2t \rangle$, $0 \le t \le 1$
*linear



x	y	z	
2	2	4	$t=0$
3	5	2	$t=1$

better drawing of c



Example: Show that the curve $\mathbf{r}(t) = \left\langle \sin(t), 2\cos(t), \sqrt{3}\sin(t) \right\rangle$ lies on both a plane and a sphere. What does the space curve for $\mathbf{r}(t)$ look like?

$x = \sin t \qquad y = 2\cos t \qquad z = \sqrt{3}\sin t$

$z = \sqrt{3}\,x$

sphere

$x^2 + y^2 + z^2$

$= \sin^2 t + 4\cos^2 t + 3\sin^2 t$

$= 4\sin^2 t + 4\cos^2 t$

$= 4(\sin^2 t + \cos^2 t)$

$= 4$

$x^2 + y^2 + z^2 = 4$

see graph online

Example: Find a vector function that represents the curve of intersection of the two surfaces. $x^2 + y^2 = 4$ and $z = xy$

$x = 2\cos\theta$

$y = 2\sin\theta$

$z = 4\cos\theta\sin\theta$

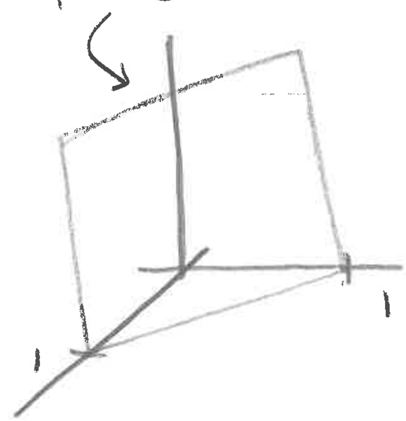
$r(\theta) = \langle 2\cos\theta, 2\sin\theta, 4\cos\theta\sin\theta \rangle$

Example: Sketch the curve $x = \cos^2 t$, $y = \sin^2 t$, and $z = t$.

$$\cos^2 t + \sin^2 t = 1$$

$$x + y = 1$$

plane



$z = t$ so move up along plane & z as t increases

*can find a few points to help graph

$t = 0$ $(1, 0, 0)$

*compare to $x = \cos t$ $y = \sin t$ $z = t$

instead of a plane will get a sphere

$$\cos^2 t + \sin^2 t = 1$$

$$x^2 + y^2 = 1$$