In Calculus 2, the length of a two-dimensional smooth curve that is only traversed once on an interval I was given by:

$$L = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

This can be extended to a space curve. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ on the interval $a \le t \le b$, then the length of the curve is given by:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt \quad \text{or} \quad L(t) = \int_a^b |r'(t)|\, dt$$

A curve $\mathbf{r}(t)$ is called smooth on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq 0$ on the interval. A smooth curve has no sharp corners or cusps → the tangent vector has continuous movement.

1. Find the length of the arc for $\mathbf{r}(t) = \langle 3t, 2\sin(t), 2\cos(t)\rangle$ from the point $\underbrace{(0, 0, 2)}_{t=0}$ to $\underbrace{(6\pi, 0, 2)}_{t=2\pi}$.

$$L = \int_0^{2\pi} \sqrt{(3)^2 + (2\cos t)^2 + (-2\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9 + 4\cos^2 t + 4\sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{9 + 4(\cos^2 t + \sin^2 t)}\, dt = \int_0^{2\pi} \sqrt{9+4}\, dt = \sqrt{13}\, t \Big|_0^{2\pi} = \boxed{2\pi\sqrt{13}}$$

Arc Length Function, s, is

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

The arc length s is called the arc length parameter

$\star$ $\frac{ds}{dt} = |r'(t)|$

Speed at time t $= \frac{ds}{dt} = |\mathbf{r}'(t)|$

$$\frac{d}{dt}(e^t \sin t) = e^t \sin t + e^t \cos t$$
$$= e^t(\sin t + \cos t)$$

2. Find the arc length function for $\mathbf{r}(t) = \langle e^t, e^t \sin(t), e^t \cos(t)\rangle$ from the point $\underbrace{(1, 0, 1)}_{t=0}$ in the direction of increasing t.

$$r'(t) = \langle e^t, e^t(\cos t + \sin t), e^t(\cos t - \sin t)\rangle$$

$$s(t) = \int_0^t \sqrt{(e^u)^2 + (e^u(\cos u + \sin u))^2 + (e^u(\cos u - \sin u))^2}\, dt$$

$$= \int_0^t \sqrt{e^{2u} + e^{2u}(\cos^2 u + 2\cos u \sin u + \sin^2 u) + e^{2u}(\cos^2 u - 2\cos u \sin u + \sin^2 u)}\, dt$$

$$= \int_0^t \sqrt{e^{2u} + 2e^{2u}\cos^2 u + 2e^{2u}\sin^2 u}\, dt$$

$$= \int_0^t \sqrt{e^{2u} + 2e^{2u}(\cos^2 u + \sin^2 u)}\, dt = \int_0^t \sqrt{3e^{2u}}\, dt = \int_0^t \sqrt{3}\, e^u\, dt$$

$$= \sqrt{3}\, e^u \Big|_0^t$$

$$= \sqrt{3}\, e^t - \sqrt{3}$$

$$\boxed{s(t) = e^t\sqrt{3} - \sqrt{3}}$$

3. Reparametrize the curve $\mathbf{r}(t) = \langle 1 + 2t, 3 + t, -5t\rangle$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$$r'(t) = \langle 2, 1, -5\rangle$$

$$s(t) = \int_0^t \sqrt{(2)^2 + (1)^2 + (-5)^2}\, du$$

$$= \int_0^t \sqrt{30}\, du$$

$$= \sqrt{30}\, u \Big|_0^t$$

$$= \sqrt{30}\, t$$

$$s(t) = \sqrt{30}\, t$$

$$t = \frac{s(t)}{\sqrt{30}} = \frac{s}{\sqrt{30}}$$

★ s is arc length function

redifine:

$$r(s) = \langle 1 + 2(s/\sqrt{30}), 3 + s/\sqrt{30}, -5(s/\sqrt{30})\rangle$$

$$= \left\langle 1 + \frac{2s}{\sqrt{30}}, 3 + \frac{s}{\sqrt{30}}, -\frac{5s}{\sqrt{30}} \right\rangle$$

4. Find the arc length parametrization of the helix $\mathbf{r}(t) = \langle \cos(4t), \sin(4t), 3t \rangle$

$$r'(t) = \langle -4\sin 4t, \quad 4\cos 4t, \quad 3 \rangle$$

$$s(t) = \int_0^t \sqrt{(-4\sin 4u)^2 + (4\cos 4u)^2 + 9} \; du$$

$$= \int_0^t \sqrt{16\sin^2 4u + 16\cos^2 4u + 9} \; du$$

$$= \int_0^t \sqrt{16(\sin^2 4u + \cos^2 4u) + 9} \; du$$

$$= \int_0^t \sqrt{16+9} \; du$$

$$= 5u \Big|_0^t$$

$$= 5t$$

$$s = 5t$$

$$t = \frac{s}{5}$$

$$r(s) = \langle \cos(4s/5), \quad \sin(4s/5), \quad 3s/5 \rangle$$