*locally linear - if its graph looks flatter and flatter as we zoom in on point $P(a, b, f(a,b))$

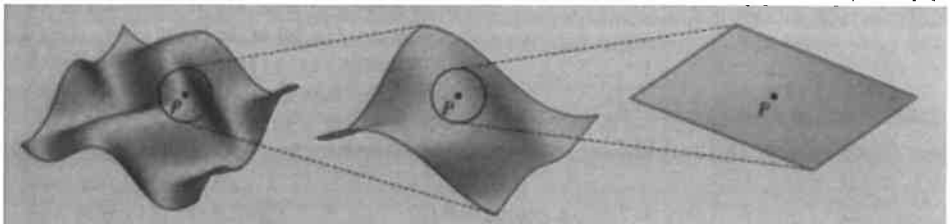

FIGURE 2 The graph looks flatter and flatter as we zoom in on a point P.

Equation of the Tangent Plane

If $f(x,y)$ is locally linear at (a,b), then its tangent plane is given by the equation

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

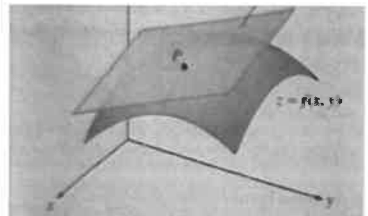

FIGURE 1 Tangent plane to the graph of $z = f(x, y)$.

1. Given that $f(x,y) = 5x + 4y^2$ is differentiable. Find the equation of the tangent plane at $(2,1)$.

$f_x(x,y) = 5$

$f_x(2,1) = 5$

$f_y(x,y) = 8y$

$f_y(2,1) = 8(1) = 8$

$f(2,1) = 5(2) + 4(1)^2$

$= 10 + 4$

$= 14$

$z = 14 + 5(x-2) + 8(y-1)$

$= 14 + 5x - 10 + 8y - 8$

$\boxed{\text{tangent plane through } (2,1,14) \quad z = 5x + 8y - 4}$

2. Find a tangent plane of the graph of $f(x,y) = xy^3 + x^2$ at $(2,-2)$.

$f_x(x,y) = y^3 + 2x$

$f_x(2,-2) = -8 + 4$

$= -4$

$f_y(x,y) = 3xy^2$

$f_y(2,-2) = 3(2)(4)$

$= 24$

$f(2,-2) = 2(-8) + 4$

$= -12$

$z = -12 - 4(x-2) + 24(y+2)$

$z = -12 - 4x + 8 + 24y + 48$

$\boxed{z = 44 - 4x + 24y}$

3. What is the equation of the tangent plane at (1,1) to the surface $4 - x^2 - y^2 = z$.

$f_x(x,y) = -2x$ $\quad$ $f_y(x,y) = -2y$ $\quad$ $f(1,1) = 4-1-1$

$f_x(1,1) = -2$ $\quad$ $f_y(1,1) = -2$ $\quad$ $= 2$

$z = 2 - 2(x-1) - 2(y-1)$

$= 2 - 2x + 2 - 2y + 2$

$z = 6 - 2x - 2y$

a. Estimate $f(1.1, 0.9)$ given $f(x,y) = z$

$f(1.1, 0.9) \approx 6 - 2(1.1) - 2(0.9)$

$= 6 - 2.2 - 1.8$

$= 2$

Recall a tangent line in 2d:

$$y = y_0 + m(x - x_0)$$

This idea continues into multiple variables and dimensions so long as it is locally linear. Try to write an equation that represents a linear approximation in 3d.

$$z \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

We can also write the linear approximation in term of the change in f: $\Delta f = f(x,y) - f(a,b)$

* subtracting $f(a,b)$ over

$$\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

$$\Delta f \approx df, \qquad \Delta x \approx dx, \qquad \Delta y \approx dy$$

$$df = \Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y = \boxed{\frac{df}{dx}dx + \frac{df}{dy}dy}$$

differential or total differential

4. Use the linear approximation to estimate $(3.93)^3(1.01)^4(1.98)^{-1}$

$(3.93)^3(1.01)^4(1.98)^{-1} \rightarrow x^3y^4z^{-1} = f(x,y,z)$

$f(3.93, 1.01, 1.98) \approx f(4, 1, 2)$

$f_x(x,y,z) = 3x^2y^4z^{-1}$ $\quad$ $f_y(x,y,z) = 4x^3y^3z^{-1}$

$f_x(4,1,2) = 3(16)(1)(1/2)$ $\quad$ $f_y(4,1,2) = 4(64)(1)(1/2)$

$= 24$ $\quad$ $= 128$

$f_z(x,y,z) = -x^3 y^4 z^{-2}$

$f(x,y,z) = 64(1)(1/2)$

$= 32$

$f_z(4,1,2) = -64(1)(1/4)$

$= -16$

tangent plane of $f(x,y,z)$ through $(4,1,2)$ $= 32 + 24(x-4) + 128(y-1) - 16(z-2)$

$f(3.93, 1.01, 1.98) \approx 32 + 24(3.93-4) + 128(1.01-1) - 16(1.98-2)$

$= 32 + 24(-0.07) + 128(0.01) - 16(-0.02)$

$= 32 - 1.68 + 1.28 + 0.32$

$= 31.92$

5. Use differentials to find an approximate value for $\sqrt{1.03^2 + 1.98^3}$

$\sqrt{1.03^2 + 1.98^3} \rightarrow f(x,y) = \sqrt{x^2+y^3}$

$f(1.03, 1.98) \approx f(1,2)$

$f_x(x,y) = \frac{1}{2}(x^2+y^3)^{-1/2}(2x)$

$f_y(x,y) = \frac{1}{2}(x^2+y^3)^{-1/2}(3y^2)$

$f_x(1,2) = \frac{1}{2}(1+8)^{-1/2}\,2$

$= (1/6)(2) = 1/3$

$f_y(1,2) = \frac{1}{2}(1+8)^{-1/2}(12)$

$= (1/6)(12) = 2$

6. Find the total differential of $W = x^5y^3 + x^2z^4$

$dw = \frac{dw}{dx}dx + \frac{dw}{dy}dy + \frac{dw}{dz}dz$

or $W_x\,dx + W_y\,dy + W_z\,dz$

$dw = (5x^4y^3 + 2xz^4)dx + (3x^5y^2)dy + (4x^2z^3)dz$

7. Estimate the amount of material in a closed can (right circular cylinder) with a radius of 3 inches and a height of 8 inches if the material of the can is 0.04 inches thick.

$V = \pi r^2 h$

$dr = \Delta r = r - r_0$

$= -0.04$

$dh = \Delta h = h - h_0$

$= 2(-0.04)$

$= -0.08$

*top & bottom losing 0.04 in

$dV = V_r\,dr + V_h\,dh$

$V_r = 2\pi r h$

$V_r(3,8) = 2\pi(3)(8)$

$= 48\pi$

$dV = 48\pi(-0.04) + 9\pi(-0.08)$

$= -2.64\pi$

$V_h = \pi r^2$

$V_h(3,8) = 9\pi$

* $dV = V(3,8) - V(3.04, 8.08)$

#5 continued

$f(1,2) = \sqrt{1+8} = 3$

$f(x,y) = 3 + 1/3(x-1) + 2(y-2)$

$f(1.03, 1.98) \approx 3 + 1/3(1.03-1) + 2(1.98-2)$

$= 3 + 1/3(0.03) + 2(-0.02)$

$= \boxed{2.97}$

#7 a different way:

$V = \pi r^2 h$

$V_r = 2\pi r h$ $\qquad$ $V_h = \pi r^2$

$V_r(3, 8) = 2\pi(3)(8)$ $\qquad$ $V_h(3, 8) = 9\pi$

$= 48\pi$

$V = \pi(3)^2(8)$

$= 72\pi$

$V(r, h) = 72\pi + 48\pi(r-3) + 9\pi(h-8)$

$V(2.96, 7.92) \approx 72\pi + 48\pi(2.96-3) + 9\pi(7.92-8)$

$V(2.92, 7.92) - V(3, 8)$

$V(2.92, 7.92) - 72\pi \approx 48\pi(-0.04) + 9\pi(-0.08)$

$= -2.64\pi$