https://www.youtube.com/watch?v=GJODOGq7cAY&list=PLHXZ9OQGMqxc_CvEy7xBKRQr6I214QJcd&index=18

Directional Derivative if $\vec{u}$ is a unit vector then

$$D_{\vec{u}} f(P) = \nabla f_P \cdot \vec{u}$$

1. Find the directional derivative of $f(x,y) = 2 - x^2 - y^2$ at $(\frac{1}{2}, -\frac{1}{2})$ in the direction of $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ $\quad \|\vec{u}\| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

$\vec{u}$ already a unit vector

$f_x = -2x \qquad f_y = -2y$

$\nabla f = \langle -2x, -2y \rangle$

$\nabla f_{(1/2, -1/2)} = \langle -1, 1 \rangle$

$D_{\vec{u}} f(\frac{1}{2}, -\frac{1}{2}) = \langle -1, 1 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \boxed{0}$

Consider the angle the unit vector $\vec{u}$ makes. $\vec{u} = \langle \cos\theta, \sin\theta \rangle$

$\cos\theta = \frac{x}{1} \qquad \sin\theta = \frac{y}{1} \qquad \vec{u} = \langle x, y \rangle$

2. Let $f(x,y) = xe^y$, at the point $(2, -1)$ find the directional derivative in the direction of $\vec{v} = \langle 2, 3 \rangle$.

$\vec{u} = \frac{\langle 2, 3 \rangle}{\sqrt{4+9}} = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle$

$f_x = e^y \qquad f_y = xe^y$

$\nabla f = \langle e^y, xe^y \rangle$

$\nabla f_{(2,-1)} = \langle \frac{1}{e}, \frac{2}{e} \rangle$

$D_{\vec{u}} f(2,-1) = \langle \frac{1}{e}, \frac{2}{e} \rangle \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle$

$= \frac{2}{e\sqrt{13}} + \frac{6}{e\sqrt{13}}$

$= \frac{8}{e\sqrt{13}} \approx \boxed{0.82}$

3. Find the rate of change of $f(x, y) = x^2 + 2xy - 3y^2$ at the point $(1, 2)$ in the direction indicated by the angle $\theta = \frac{\pi}{4}$. $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ unit vector already

$\nabla f = \langle 2x + 2y,\ 2x - 6y \rangle$

$\nabla f(1,2) = \langle 6, -10 \rangle$

$D_{\vec{u}} f(1,2) = \langle 6, -10 \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

$= 6/\sqrt{2} - 10/\sqrt{2} = \boxed{-4/\sqrt{2}}$

4. Find the gradient and the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\vec{v} = \langle 2, 5 \rangle$

$\vec{u} = \frac{\langle 2,5 \rangle}{\sqrt{4+25}} = \langle 2/\sqrt{29}, 5/\sqrt{29} \rangle$

$\nabla f = \langle 2xy^3,\ 3x^2y^2 - 4 \rangle$

$\nabla f(2, -1) = \langle -4, 8 \rangle$

$D_{\vec{u}} f(2,-1) = \langle -4, 8 \rangle \cdot \langle 2/\sqrt{29}, 5/\sqrt{29} \rangle$

$= -\frac{8}{\sqrt{29}} + \frac{40}{\sqrt{29}} = \boxed{\frac{32}{\sqrt{29}}}$

5. Find the directional derivative of the function $f(x, y, z) = z^4 - x^3y^2$ at the point $P(1, 3, 2)$ in the direction of point $Q(2, 4, 3)$.

$\nabla f = \langle -3x^2y^2,\ -2x^3y,\ 4z^3 \rangle$

$\vec{PQ} = \langle 1, 1, 1 \rangle$

$\nabla f(1,3,2) = \langle -27, -6, 32 \rangle$

$\vec{u} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}}$

$= \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$

$D_u f(1,3,2) = \langle -27, -6, 32 \rangle \cdot \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$

$= -\frac{27}{\sqrt{3}} + \frac{-6}{\sqrt{3}} + \frac{32}{\sqrt{3}}$

$= \boxed{-\frac{1}{\sqrt{3}}}$