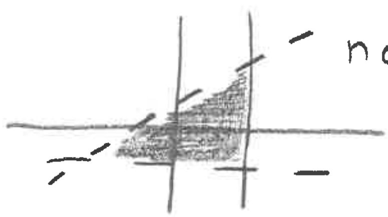
Absolute Maximum and Absolute Minimum

Definition: A function f has a **absolute maximum** at (a, b) if $f(a, b)$ is the largest function value for the domain of f. Similarly, f has a **absolute minimum** at (a, b) if $f(a, b)$ is the smallest function value for the domain of f.

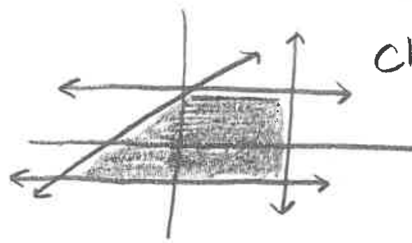
Extreme Value Theorem for Functions of One Variable: If f is continous on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum value. These are found by evaluating critical points and the endpoints of the interval.

Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed and bounded set D in $\Re^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Definition: A **closed set** in $\Re^2$ is one that contains all of its boundry points.



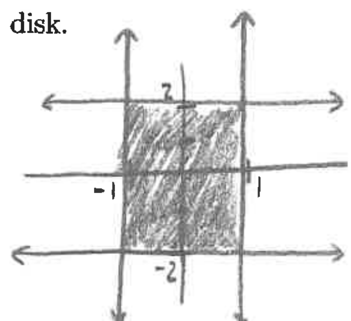
not closed



Closed

Definition: A **bounded set** in $\Re^2$ is one that is contained in some disk.

$D = \{(x,y) \mid |x| \le 1, |y| \le 2\}$

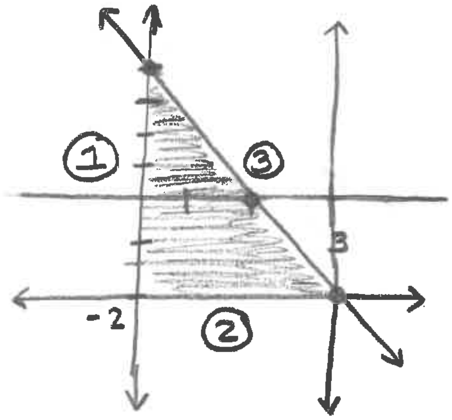


To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(1) Find the values of f at the critical points in D. $f_x = 0$ and $f_y = 0$
(2) Find the extreme values of f on the boundry of D.
(3) The largest of the values is the absolute maximum value; the smallest is the absolute minimum value.

Example: Find the absolute maximum/absolute minimum of f on the set D.

$D = \{(x,y) \mid 0 \le x \le 3,\ \ -2 \le y \le 4 - 2x\}$



$f(x,y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10$

$= 4x^2 + 4xy + 8y^2 - 12x + 8y + 10$

$f_x = 8x + 4y - 12$ $\qquad$ $f_y = 4x + 16y + 8$

$0 = 8x + 4y - 12$ $\qquad$ $0 = 4x + 16y + 8$

system of equations

$$\begin{aligned} 8x + 4y &= 12 \\ 4x + 16y &= -8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 8x + 4y &= 12 \\ -8x - 32y &= 16 \\ \hline -28y &= 28 \\ y &= -1 \end{aligned}$$

$8x + 4(-1) = 12$

$8x = 16$

$x = 2$

critical point $(2,-1)$

<u>side 1</u>

$x = 0 \quad -2 \le y \le 4$

$f(0,y) = 4(0)^2 + 4(0)y + 8y^2 - 12(0) + 8y + 10$

$= 8y^2 + 8y + 10$

$f'(0,y) = 16y + 8$

$0 = 16y + 8$

$y = -\frac{1}{2}$

point $(0, -\frac{1}{2})$

side 2

$y = -2 \quad 0 \le x \le 3$

$f(x, -2) = 4x^2 + 4x(-2) + 8(-2)^2 - 12x + 8(-2) + 10$

$= 4x^2 - 8x + 32 - 12x - 16 + 10$

$= 4x^2 - 20x + 26$

$f'(x, -2) = 8x - 20$

$0 = 8x - 20$

$x = 5/2$

point $(5/2, -2)$

side 3

$0 \le x \le 3 \quad y = 4 - 2x$

$f(x, 4-2x) = 4x^2 + 4x(4-2x) + 8(4-2x)^2 - 12x + 8(4-2x) + 10$

$= 4x^2 + 16x - 8x^2 + 8(16 - 16x + 4x^2) - 12x + 32 - 16x + 10$

$= 4x^2 + 16x - 8x^2 + 128 - 128x + 32x^2 - 12x + 32 - 16x + 10$

$= 28x^2 - 140x + 170$

$f'(x, 4-2x) = 56x - 140$

$0 = 56x - 140$

$x = 5/2$

$y = 4 - 2(5/2)$
$= -1$

point
$(5/2, -1)$

corners

$(0, 4)$

$(0, -2)$

$(3, -2)$

	critical points	$f(a, b)$	
regular c.p.	$(2, -1)$	-6	absolute min
corners	$(0, 4)$	26	
	$(0, -2)$	170	absolute max
	$(3, -2)$	2	
side c.p.	$(0, -1/2)$	8	
	$(5/2, -2)$	1	
	$(5/2, -1)$	25	

or can solve for y to graph

Example: Find the absolute max for $f(x,y) = xy$ on the set D.

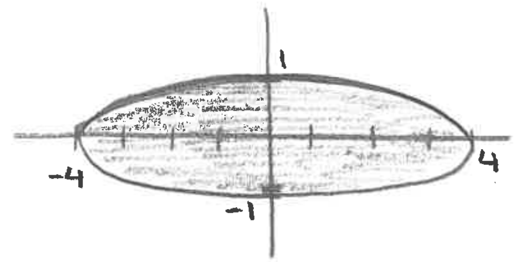
$$y = \pm\sqrt{1 - x^2/16}$$

$$D = \left\{ (x,y) \,\middle|\, \frac{x^2}{16} + y^2 \le 1 \right\}$$

$$\frac{x^2}{16} + y^2 \le 1$$

ellipse

$\leftarrow 4 \rightarrow$



$f_x = y \qquad f_y = x$

$0 = y \qquad 0 = x$

critical point $(0,0)$

* Divide "sides" into arcs

$y = \sqrt{1 - x^2/16}$ and $y = -\sqrt{1 - x^2/16}$

arc 1

$$f\left(x, \sqrt{1 - x^2/16}\right) = x\sqrt{1 - x^2/16}$$

$$f'\left(x, \sqrt{1 - x^2/16}\right) = \sqrt{1 - x^2/16} + x\left(\tfrac{1}{2}\left(1 - x^2/16\right)^{-1/2}\left(-\tfrac{2}{16}x\right)\right)$$

$$= \sqrt{1 - x^2/16} - \frac{x^2}{16\sqrt{1 - x^2/16}}$$

$$= \frac{16\left(1 - x^2/16\right) - x^2}{16\sqrt{1 - x^2/16}}$$

$$0 = \frac{16 - 2x^2}{16\sqrt{1 - x^2/16}}$$

$x = \pm\sqrt{8} \qquad y = \frac{1}{\sqrt{2}}$

$(\sqrt{8}, 1/\sqrt{2})$ and $(-\sqrt{8}, 1/\sqrt{2})$

arc 2

$$f\left(x, -\sqrt{1-x^2/16}\right) = -x\sqrt{1-x^2/16}$$

$$f'\left(x, -\sqrt{1-x^2/16}\right) = -\sqrt{1-x^2/16} - x\left(\frac{1}{2}\left(1-x^2/16\right)^{-1/2}\left(-{}^2/_{16}\,x\right)\right)$$

$$= -\sqrt{1-x^2/16} + \frac{x^2}{16\sqrt{1-x^2/16}}$$

$$= \frac{-16\left(1-x^2/16\right) + x^2}{16\sqrt{1-x^2/16}}$$

$$0 = \frac{-16 + 2x^2}{16\sqrt{1-x^2/16}}$$

$x = \pm\sqrt{8}$ $\qquad$ $y = -\frac{1}{\sqrt{2}}$

$(\sqrt{8}, -1/\sqrt{2})$ and $(-\sqrt{8}, -1/\sqrt{2})$

critical points	$f(x,y)$
$(0,0)$	0
$(\sqrt{8}, 1/\sqrt{2})$	2
$(-\sqrt{8}, 1/\sqrt{2})$	-2
$(\sqrt{8}, -1/\sqrt{2})$	-2
$(-\sqrt{8}, -1/\sqrt{2})$	2

absolute max $= 2$