Section 14.7: Maximum and Minimum values

Definition: A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a **local maximum value.** If $f(x, y) \geq f(a, b)$ for all (x, y) in such disk, $f(a, b)$ is a **local minimum value.**

Note: The word local is sometimes replaced with the word relative.

Theorem: If f has a **local extremum** (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives of f exists there, then $\boxed{f_x(a, b) = 0 \text{ and } f_y(a, b) = 0}$

Note: If the graph of f has a tangent plane at a local extremum, then the tangent plane is horizontal.

Definition: A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of these partial derivatives does not exist, is called a **critical point** of f.

Second Derivative Test: Suppose the second partial derivatives of f are continuous in a disk with center (a, b), and suppose that (a, b) is a critical point of f. Let

determinant

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx} > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx} < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is a sadle point.
(d) If $D = 0$ then the test gives no information.

$$\text{determinant} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx}$$

Example: Find and classify the critical values of $f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$

$$f_x = -6x^2 - 12x + 48$$
$$0 = -6(x^2 + 2x - 8)$$
$$= (x+4)(x-2)$$
$$x = 2, -4$$

$$f_y = 3y^2 - 12y$$
$$0 = 3y(y-4)$$
$$y = 0, 4$$

$$f_{xx} = -12x - 12$$
$$f_{yy} = 6y - 12$$
$$f_{xy} = 0$$

critical points	$D = f_{xx} f_{yy} - (f_{xy})^2$	f_{xx}	conclusion
$(2,0)$	$(-36)(-12) - 0 > 0$	neg	local max
$(2,4)$	$(-36)(12) - 0 < 0$	doesn't matter $(D<0)$	saddle point
$(-4,0)$	$36(-12) - 0 < 0$	doesn't matter	saddle point
$(-4,4)$	$36(12) - 0 > 0$	pos	local min

Example: Find and classify the critical values of $f(x, y) = x^3 + 6xy - 2y^2$

$f_x = 3x^2 + 6y$

$0 = 3x^2 + 6y$

$f_y = 6x - 4y$

$0 = 6x - 4y$

$\frac{-6x}{-4} = y$

$\frac{3}{2}x = y$

$f_{xx} = 6x$

$f_{yy} = -4$

$f_{xy} = 6$

$0 = 3x^2 + 6(3/2 x)$

$0 = 3x^2 + 9x$

$= 3x(x+3)$

$x = 0, -3$

$x = 0 \quad y = 0$

$x = -3 \quad y = -9/2$

critical points	$D = f_{xx}f_{yy} - (f_{xy})^2$	f_{xx}	conclusion
$(0,0)$	$0(-4) - 6^2 < 0$		saddle point
$(-3, -9/2)$	$(-18)(-4) - 6^2 > 0$	neg	local max

Example: Find and classify the critical values of $f(x, y) = 1 + 2xy - x^2 - y^2$

$f_x = 2y - 2x$

$0 = 2y - 2x$

$x = y$

$f_y = 2x - 2y$

$0 = 2x - 2y$

$\frac{-2x}{-2} = y$

$x = y$

$f_{xx} = -2$

$f_{yy} = -2$

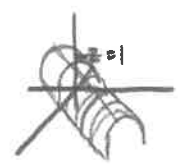
$f_{xy} = 2$

$D = (-2)(-2) - 2^2$

$= 0$

no information

can check graphically



$f(x,y) = 1 + 2xy - x^2 - y^2$

$f(x,y) = 1 - [x^2 - 2xy + y^2]$

$f(x,y) = 1 - (x-y)^2$

when $x = y$

$f(x,y) = 1$