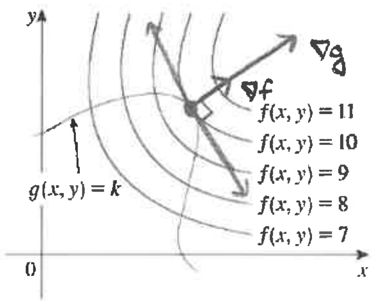
Section 14.8: Lagrange Multipliers

Lagrange multipliers are another method used to maximizing or minimizing a general function $f(x,y)$ subject to a constraint $g(x,y) = k$.

We are trying to find the extreme values of $f(x,y)$ subject to the condition $g(x,y) = k$. Thus we want to find the point(s) (a,b) on $g(x,y) = k$ so that the value of $f(a,b)$ will be a maximum or minimum.



∇g & ∇f perpendicular to tangent line (normal)

* in same direction & differ only by a scalar → λ

Thus we want $f(x,y) = c$ some constant to just touch $g(x,y) = k$. This happens when both level curves have a common tangent line. This means that the normal lines where they touch, at the point (x_o, y_o) are identical. Thus

$\boxed{\nabla f(x_o, y_o) = \lambda \nabla g(x_o, y_o)}$ for some scalar λ.

The number λ is called a **Lagrange multiplier**.

Method of Lagrange multipliers: To find the maximum and minimum values of $f(x,y,z)$ subject to the constraints $g(x,y,z) = k$:

(a) Find all values of x, y, z, and λ such that

$\nabla f = \lambda \nabla g$ and $g(x,y,z) = k$

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

(b) Evaluate f at all points (x,y,z) that arise from step (a). The largest of these values is the maximum value of f and the smallest is the minimum value of f.

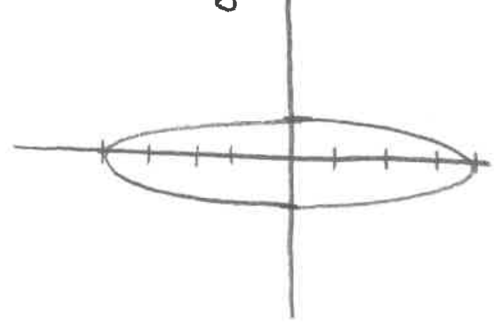
Note: If there are two constraints, g and h, then $\nabla f = \lambda \nabla g + \mu \nabla h$

$x^2+16y^2-16=0$

Example: Find the extreme values of the function $f(x,y)=x^2+2y^2$ on the ellipse $x^2+16y^2=16$

constraint $g(x,y)=x^2+16y^2-16$

$\nabla f=\langle 2x, 4y\rangle \qquad \nabla g=\langle 2x, 32y\rangle$



$2x=\lambda 2x \qquad 4y=\lambda 32y$

$2x-\lambda 2x=0 \qquad 4y-\lambda 32y=0$

$2x(1-\lambda)=0 \qquad 4y(1-8\lambda)=0$

$x=0$ or $\lambda=1 \qquad y=0$ or $\lambda=1/8$

when $x=0$

$0^2+16y^2=16$

$y^2=\pm 1$

points	$f(x,y)$	
$(0,1)$	2	min
$(0,-1)$	2	
$(-4,0)$	16	max
$(4,0)$	16	

when $\lambda=1$

$4y=32y$

$0=28y$

$y=0$

when $y=0$

$x^2+16(0)^2=16$

$x=\pm 4$

when $\lambda=1/8$

$2x=1/8(2x)$

$2x=1/4\, x$

$8x=x$

$7x=0$

$x=0$

Example: Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum value.

constraint $g(x,y,z) = x^2+y^2+z^2-25$

$\nabla f = \langle 1, 2, 3 \rangle \qquad \nabla g = \langle 2x, 2y, 2z \rangle$

$1 = \lambda 2x \qquad 2 = \lambda 2y \qquad 3 = \lambda 2z$

$\frac{1}{2x} = \lambda \qquad \frac{1}{y} = \lambda \qquad \frac{3}{2z} = \lambda$

* 3 variable system of eqts

$\frac{1}{2x} = \frac{1}{y} \qquad \frac{1}{2x} = \frac{3}{2z}$

$2x = y \qquad z = 3x$

plug into constraint

$x^2 + (2x)^2 + (3x^2) = 25$

$x^2 + 4x^2 + 9x^2 = 25$

$14x^2 = 25$

$x = \pm \frac{5}{\sqrt{14}}$

points	$f(x,y,z)$	
$(5/\sqrt{14}, 10/\sqrt{14}, 15/\sqrt{14})$	$\frac{40\sqrt{14}}{7}$	max
$(-5/\sqrt{14}, -10/\sqrt{14}, -15/\sqrt{14})$	$-\frac{40\sqrt{14}}{7}$	min

when $x = 5/\sqrt{14}$
$y = 10/\sqrt{14}$
$z = 15/\sqrt{14}$

when $x = -5/\sqrt{14}$
$y = -10/\sqrt{14}$
$z = -15/\sqrt{14}$

Example: Find the point on the sphere $x^2 + y^2 + z^2 = 9$ that is closest to the point $(2, 3, 4)$.

* Calc I optimization *

constraint $d = \sqrt{(x-2)^2 + (y-3)^2 + (z-4)^2}$

recall $\sqrt{\ }$ doesn't effect answer

$d = (x-2)^2 + (y-3)^2 + (z-4)^2$

$g(x, y, z) = (x-2)^2 + (y-3)^2 + (z-4)^2$

$f(x, y, z) = x^2 + y^2 + z^2 - 9$

$\nabla f = \langle 2x, 2y, 2z \rangle$ $\quad\quad$ $\nabla g = \langle 2x-4, 2y-6, 2z-8 \rangle$

$2x = \lambda(2x-4)$ $\quad\quad$ $2y = \lambda(2y-6)$ $\quad\quad$ $2z = \lambda(2z-8)$

$x = \lambda(x-2)$ $\quad\quad$ $y = \lambda(y-3)$ $\quad\quad$ $z = \lambda(z-4)$

$\frac{x}{x-2} = \lambda$ $\quad\quad$ $\frac{y}{y-3} = \lambda$ $\quad\quad$ $\frac{z}{z-4} = \lambda$

$\frac{x}{x-2} = \frac{y}{y-3}$ $\quad\quad$ $\frac{x}{x-2} = \frac{z}{z-4}$

$x(y-3) = y(x-2)$ $\quad\quad$ $x(z-4) = z(x-2)$

$xy - 3x = xy - 2y$ $\quad\quad$ $zx - 4x = zx - 2z$

$-3x = -2y$ $\quad\quad$ $-4x = -2z$

$\frac{3x}{2} = y$ $\quad\quad$ $2x = z$

take $x^2 + y^2 + z^2 = 9$

$$x^2 + (3x/2)^2 + (2x)^2 = 9$$

$$x^2 + \frac{9x^2}{4} + 4x^2 = 9$$

$$\frac{29x^2}{4} = 9$$

$$x^2 = \frac{36}{29}$$

$$x = \pm \frac{6}{\sqrt{29}}$$

$x = 6/\sqrt{29}$

$y = 9/\sqrt{29}$

$z = 12/\sqrt{29}$

$x = -6/\sqrt{29}$

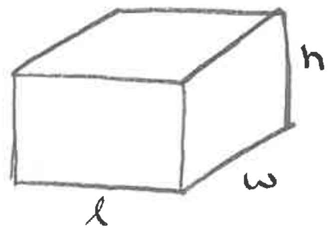
$y = -9/\sqrt{29}$

$z = 12/\sqrt{29}$

points	distance
$(6/\sqrt{29}, 9/\sqrt{29}, 12/\sqrt{29})$	2.38516
$(-6/\sqrt{29}, -9/\sqrt{29}, -12/\sqrt{29})$	8.38516

$(6/\sqrt{29}, 9/\sqrt{29}, 12/\sqrt{29})$ closest

Example: The base of a rectangular tank with volume of 540 cubic units is made of slate and the sides are made of glass. If slate costs five times as much as glass(per unit area), find the dimensions of the tank which minimize the cost of the materials.



$V = lwh$

$540 = lwh$

$V = 540$

$V(l, w, h) = lwh$

$SA = lw + 2lh + 2wh$

$C = 5lw + 2lh + 2wh$

base = 5 sides
(per unit area)

$\nabla V = \langle wh,\ lh,\ lw \rangle$

$\nabla C = \langle 5w + 2h,\ 5l + 2h,\ 2l + 2w \rangle$

$5w + 2h = \lambda wh$

$\frac{5w + 2h}{wh} = \lambda$

$5l + 2h = \lambda lh$

$\frac{5l + 2h}{lh} = \lambda$

$2l + 2w = \lambda lw$

$\frac{2l + 2w}{lw} = \lambda$

$\frac{5w + 2h}{wh} = \frac{5l + 2h}{lh}$

$lh(5w + 2h) = wh(5l + 2h)$

$5lhw + 2lh^2 = 5lhw + 2wh^2$

$2lh^2 = 2wh^2$

$lh^2 = wh^2$

$l = w$

$\frac{5w + 2h}{wh} = \frac{2l + 2w}{lw}$

$lw(5w + 2h) = wh(2l + 2w)$

$5lw^2 + 2whl = 2lwh + 2hw^2$

$5lw^2 = 2hw^2$

$5l = 2h$

$h = \frac{5}{2} l$

$540 = lwh$

$540 = l(l)(5/2\, l)$

$540 = 5/2\, l^3$

$216 = l^3$

$6 = l$

$w = 6$

$h = 5/2\,(6) = 15$

$\boxed{6 \text{ by } 6 \text{ by } 15}$