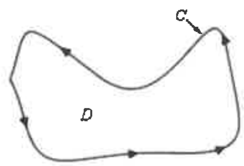
Green's Theorem
Multivariable Calculus

Take the simple (doesn't intersect) closed curve C below and let D be the region enclosed by the curve.



Positive Orientation: The direction placed on the curve is in the counterclockwise direction. (as the curve is traced region D is always on the left)

Green's Theorem

Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on D then,

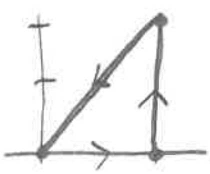
$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Alternate notations:

$$\oint_C P dx + Q dy$$

check conditions

1. Use Green's Theorem to evaluate $\oint_C xy\, dx + x^2 y^3 dy$ where C is the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 2)$ with positive orientation



$P = xy$ $\quad Q = x^2y^3$

$\frac{\partial P}{\partial y} = x$ $\quad \frac{\partial Q}{\partial x} = 2xy^3$

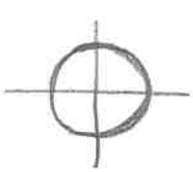
$$\int_0^1 \int_0^{2x} (2xy^3 - x)\, dy\, dx$$

$$= \boxed{\frac{2}{3}}$$

$0 \le x \le 1$

$0 \le y \le 2x$

2. Evaluate $\oint_C y^3\, dx - x^3 dy$ where C is the positively oriented circle of radius 2 centered at the origin.



$P = y^3$ $\quad Q = -x^3$

$\partial P = 3y^2$ $\quad \partial Q = -3x^2$

$$\iint_D (-3x^2 - 3y^2)\, dA$$

$$-3 \iint_D (x^2 + y^2)\, dA$$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$

convert to polar

$$-3\int_0^{2\pi}\int_0^{2} r^2 \cdot \underbrace{r\, dr\, d\theta}_{\text{Jacobian}} = \boxed{-24\pi}$$