Derivative of a Function

We found the slope of a tangent line at a specific x value, a.

*Can find the slope of a tangent line at $(x,\ f(x))$, where x can represent any number in the domain of f'.

Derivative of a Function

Let $y = f(x)$ denote a function f. The derivative of f at x, denoted by $f'(x)$, read "f prime of x," is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

provided that this limit exists. The derivative of a function f gives the slope of f for any value of x is the domain of f'.

1. Find the slope of the function $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

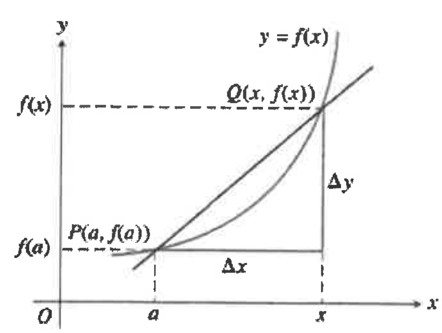
$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

Alternative Definition:

The derivative of the function f at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$



3. Use the definition $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ to find the derivative of $f(x) = 2x + 3$ at $a = -1$.

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{2x+3 - (2(-1)+3)}{x+1}$$

$$= \lim_{x \to -1} \frac{2x+3-1}{x+1}$$

$$= \lim_{x \to -1} \frac{2x+2}{x+1}$$

$$= \lim_{x \to -1} \frac{2(x+1)}{x+1}$$

$$= \lim_{x \to -1} 2$$

$$= \boxed{2}$$

Different ways to write the derivative:

1. $\frac{dy}{dx}$ "the derivative of y w/ respect to x"

2. y' "y prime"

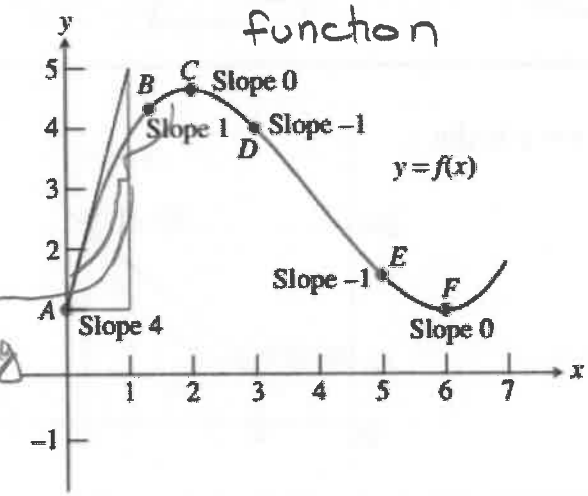
3. $\frac{df}{dx}$ "derivative of f w/ respect to x"

4. $\frac{d}{dx} f(x)$ "derivative of f at x"

Comparing graphs of derivative to function:

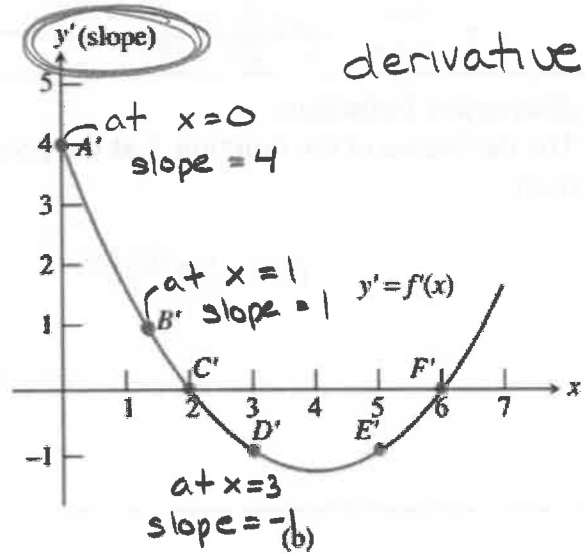
function

slope is positive but decreasing



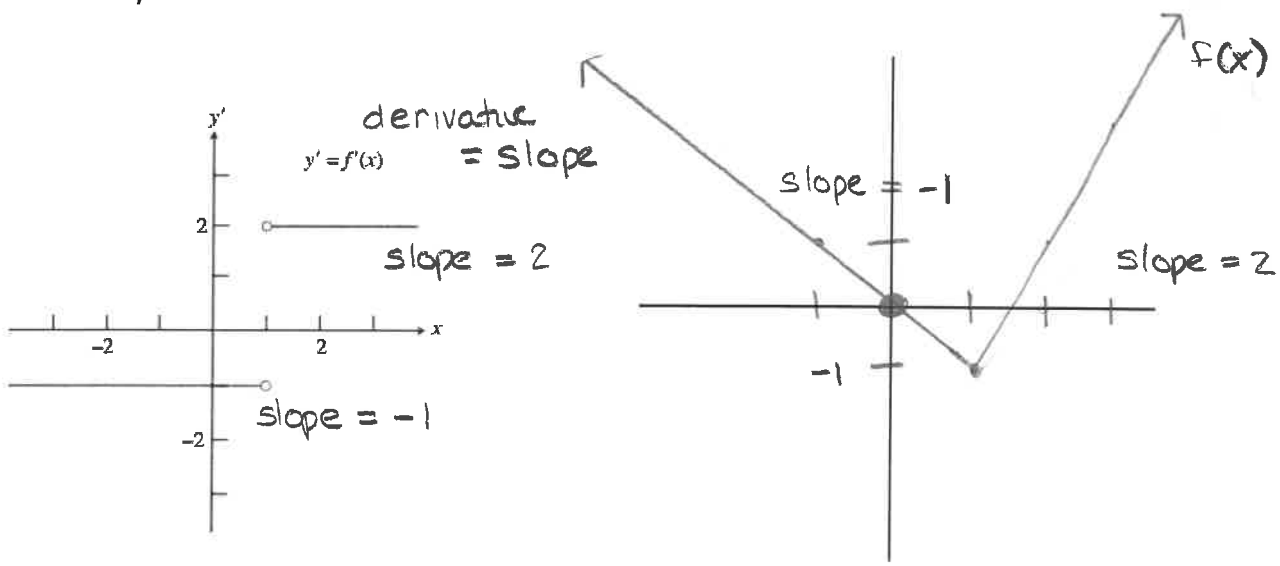
(a)

derivative

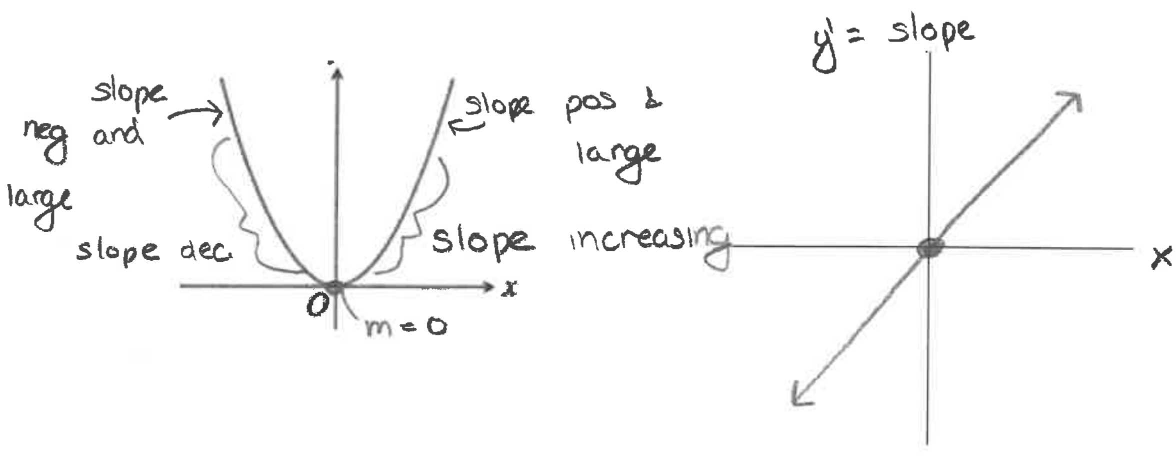


(b)

2. Sketch the graph of a function f that has the following properties:
   I. $f(0) = 0$
   II. The graph of f' is shown below
   III. f is continuous for all x



3. Sketch the graph of the derivative of the function from the function below:



One-Sided Derivatives

A function $y = f(x)$ is **differentiable on a closed interval** $[a, b]$ if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{[the right-hand derivative at } a\text{]}$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{[the left-hand derivative at } b\text{]}$$

exist at the endpoints.

4. Show that the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative at $x = 0$

$$y = \begin{cases} x^2, & x \le 0 \quad \text{left} \\ 2x, & x > 0 \quad \text{right} \end{cases}$$

left-hand derivative:

$$\lim_{h \to 0^-} \frac{(0+h)^2 - 0^2}{h}$$

$$= \lim_{h \to 0^-} \frac{h^2}{h}$$

$$\lim_{h \to 0^-} h = 0$$

right-hand derivative

$$\lim_{h \to 0^+} \frac{2(0+h) - 2(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{2h}{h}$$

$$= \lim_{h \to 0^+} 2 = 2$$

left hand & right hand deriv. are different

$0 \neq 2$

so not differentiable at $x = 0$

5. Find the unique value of k that makes the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ 3x + k, & x > 1 \end{cases}$$

differentiable at $x = 1$.

$$f'(x) = \begin{cases} 3x^2, & x \le 1 \\ 3, & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f'(x) = 3(1)^2$$
$$= 3$$

$$\lim_{x \to 1^+} f'(x) = 3$$

Also must be continuous!

$$\lim_{x \to 1^-} f(x) = 1^3$$

$$\lim_{x \to 1^+} f(x) = 3 + k$$

$$1 = 3 + k$$

$$\boxed{-2 = k}$$