RULE 1 Derivative of a Constant Function

If f is the function with the constant value c, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

1. Find $\frac{dy}{dx}$

   a. $f(x) = 4$

$$\frac{dy}{dx} = f'(x) = 0$$

RULE 2 Power Rule for Positive Integer Powers of x

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

   b. $f(x) = x^2 - 1$

$$f'(x) = 2x^1$$
$$= 2x$$

RULE 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

* coefficient stays

RULE 4 The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

* same w/ limits
take derivative of each term

c. $f(x)=\frac{x^3}{3}-x$

$f(x) = \frac{1}{3}x^3 - x$

$f'(x) = \frac{1}{3}(3)x^2 - 1$

$= x^2 - 1$

d. $y=\frac{x-a}{\sqrt{x}-\sqrt{a}}$

* dont know fractions yet
↳ rationalize den

$y = \frac{x-a}{\sqrt{x}-\sqrt{a}} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

$y = \frac{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}{\cancel{x-a}}$

$y = \sqrt{x} - \sqrt{a}$

$\uparrow x^{1/2}$ $\uparrow$ constant

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

e. $f(x)=x^4-x^2+4x$

$f'(x) = 4x^3 - 2x + 4$

f. $\frac{w^3-w}{w} = y$

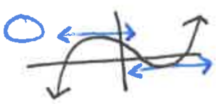
$y = \frac{w^3}{w} - \frac{w}{w}$

$y = w^2 - 1$

$y' = 2w$

2. Find the values of x for which the curve has horizontal tangents.

slope = 0



a. $y=x^3-4x^2+x+2$

$y' = 3x^2 - 8x + 1$

$0 = y'$

$0 = 3x^2 - 8x + 1$

$0 = (\quad)(\quad)$

$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(1)}}{2(3)}$

$= \frac{8 \pm \sqrt{64-12}}{6}$

$= \frac{8 \pm \sqrt{52}}{6}$

$= \frac{8 \pm 2\sqrt{13}}{6} = \frac{4 \pm \sqrt{13}}{3} \approx 0.131$ and 2.535

b. $f(x)=4x^3-6x^2-1$

$f'(x) = 12x^2 - 12x$

$0 = 12x^2 - 12x$

$0 = 12x(x-1)$

$x = 0, 1$

RULE 5 The Product Rule

The product of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

* derivative of the first, second, derivative of the second, first *

RULE 6 The Quotient Rule

At a point where $v \neq 0$, the quotient $y = u/v$ of two differentiable functions is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

* low dhigh minus high d low all over low low

RULE 7 Power Rule for Negative Integer Powers of x

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

* treat any integer power same
-neg, pos, fraction

3. Find $\frac{dy}{dx}$

a. $(x^2+1)(x^3+1) = y$

power rule

$u = x^2+1$ $\quad v = x^3+1$

$u' = 2x$ $\quad v' = 3x^2$

$y' = uv' + vu'$

$= (x^2+1)(3x^2) + (x^3+1)(2x)$

$= 3x^4 + 3x^2 + 2x^4 + 2x$

$\boxed{= 5x^4 + 3x^2 + 2x}$

* could foil original function then use power rule

b. $\dfrac{x^2+5x-1}{x^2} = y$

quotient rule

$u = x^2+5x-1$ $\quad v = x^2$

$u' = 2x+5$ $\quad v' = 2x$

$y' = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{(x^2)(2x+5) - (x^2+5x-1)(2x)}{(x^2)^2}$

$= \dfrac{2x^3 + 5x^2 - 2x^3 - 10x^2 + 2x}{x^4}$

$= \dfrac{-5x^2 + 2x}{x^4} = \boxed{\dfrac{-5x+2}{x^3}}$

c. $y = \dfrac{x}{x+1}$ $\begin{matrix} u \\ v \end{matrix}$

$$y' = \frac{v(u') - uv'}{v^2}$$

$$= \frac{(x+1)(1) - x(1)}{(x+1)^2}$$

$$= \frac{\cancel{x} + 1 - \cancel{x}}{x^2 + 2x + 1}$$

$$= \frac{1}{x^2 + 2x + 1}$$

d. $x(x+1) = y$

*watch for easier option

$$y = x^2 + x$$

$$y' = 2x + 1$$

e. $f(x) = 3x^4(2x-1)$

$$f'(x) = 3x^4(2) + (2x-1)(12x^3)$$

$$= 6x^4 + 24x^4 - 12x^3$$

$$= 30x^4 - 12x^3$$

f. $f(x) = \left(1 + \dfrac{1}{x^2}\right)(x^2+1)$

$$y' = (1 + x^{-2})(2x) + (x^2+1)(-2x^{-3})$$

$$= 2x + 2x^{-1} - 2x^{-1} - 2x^{-3}$$

$$= 2x - \frac{2}{x^3}$$

g. $f(x) = (3x-1)(2x-1)^{-1}$

$$f(x) = \frac{3x-1}{2x-1}$$

$$f'(x) = \frac{(2x-1)(3) - (3x-1)(2)}{(2x-1)^2}$$

$$= \frac{\cancel{6x} - 3 - \cancel{6x} + 2}{4x^2 - 4x + 1}$$

$$= \frac{-1}{4x^2 - 4x + 1}$$

4. Suppose u and v are functions of x that are differentiable at $x=2$ and that $u(2)=3$, $u'(2)=-3$, $v(2)=-1$, and $v'(2)=2$. Find the values of the following derivatives at $x=2$.

a. $\frac{d}{dx}(uv)$

$y' = u'v + v'u$

$y'(2) = u'(2)\,v(2) + v'(2)u(2)$

$= (-3)(-1) + (2)(3)$

$= 3 + 6$

$= 9$

c. $\frac{d}{dx}\left(\frac{v}{u}\right)$

$y' = \dfrac{uv' - vu'}{u^2}$

$y'(2) = \dfrac{3(2) - (-1)(-3)}{3^2}$

$= \dfrac{6-3}{9}$

$= \dfrac{3}{9}$

$= \dfrac{1}{3}$

b. $\frac{d}{dx}\left(\frac{u}{v}\right)$

$y' = \dfrac{vu' - uv'}{v^2}$

$y'(2) = \dfrac{(-1)(-3) - (3)(2)}{(-1)^2}$

$= \dfrac{3-6}{1}$

$= -3$

d. $\frac{d}{dx}(3u-2v+2uv)$

$y' = 3u' - 2v' + 2(u'v + v'u)$

$y'(2) = 3(-3) - 2(2) + 2((-3)(-1) + (2)(3))$

$= -9 - 4 + 2(3+6)$

$= -13 + 2(9)$

$= -13 + 18$

$= 5$

Higher Order Derivatives

The derivative $y' = \frac{dy}{dx}$ is called the 1st Derivative of y with respect to x.

*First derivative may be differentiable

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d^2y}{dx^2}$$

The derivative of the first derivative is called the second derivative of y with respect to x.

*If y'' ("y double-prime") is differentiable, its derivative

$$y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$$

The derivative of the second derivative is called the third derivative of y with respect to x.

Prime notation continues:

$$y^{(n)} = \frac{dy^{(n-1)}}{dx} = \frac{d^ny}{dx^n}$$

to denote the n^{th} derivative of y with respect to x.

5. Find the first four derivatives of the function.

a. $y = x^2 + x + 3$

$y' = 2x + 1$

$y'' = 2$

$y''' = 0$

$y'''' = 0$

b. $y = \frac{x+1}{x}$

$y' = \frac{x(1) - (x+1)(1)}{x^2}$

$= \frac{\cancel{x} - \cancel{x} - 1}{x^2}$

$= \frac{-1}{x^2}$

$y'' = -\frac{2}{x^3}$

$y''' = -\frac{6}{x^4}$

$y^{(4)} = -\frac{24}{x^5}$

Instantaneous Rate of Change

6. An orange farmer currently has 200 trees yielding and average of 15 bushels of oranges per tree. She is expanding her farm at the rate of 14 trees per year, while improved husbandry is improving her average annual yield by 1.2 bushels per tree. What is the current (instantaneous) rate of increase of her total annual production of oranges?

$t(x)$ = # of trees x years from now

$y(x)$ = yield per tree x years from now

* multiplication not addition b/c total equals

$$\underbrace{y(x) + y(x) + \ldots + y(x)}_{t(x) \text{ times}}$$

$y(x)t(x)$

$p(x) = t(x)\,y(x)$ is total production of oranges in year x

$t(0) = 200$ $\quad$ $y(0) = 15$

$t'(0) = 14$ $\quad$ $y'(0) = 1.2$

$p'(0)$ = instantaneous rate of increase currently

$$p'(0) = t'(0)\,y(0) + y'(0)\,t(0)$$

$$= 14(15) + 1.2(200)$$

$$= 210 + 240$$

$= 450$ bushels per year