1. If $f(x) = 3x^4 - 2x^3 + 5x + 2$, find $f(4)$

a. Synthetic Substitution

$$\begin{array}{r|rrrrr} 4 & 3 & -2 & 0 & 5 & 2 \\ & & 12 & 40 & 160 & 660 \\ \hline & 3 & 10 & 40 & 165 & 662 \end{array}$$

$f(4) = 662$

b. Direct Substitution

$f(4) = 3(4)^4 - 2(4)^3 + 5(4) + 2$

$= 3(256) - 2(64) + 20 + 2$

$= 768 - 128 + 22$

$= 640 + 22 = 662$ ✓

2. Given that $x + 2$ is a factor of $x^3 - 3x + 2$, find the remaining factors of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & 4 & -2 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$x^3 - 3x + 2 = (x+2)(x^2 - 2x + 1)$

$= (x+2)(x-1)(x-1)$

$= (x+2)(x-1)^2$

factors of $\dfrac{\text{constant}}{\text{leading coefficient}}$

KeyConcept Rational Zero Theorem

Words If $P(x)$ is a polynomial function with integral coefficients, then every rational zero of $P(x) = 0$ is of the form $\frac{p}{q}$, a rational number in simplest form, where p is a factor of the constant term and q is a factor of the leading coefficient.

Example Let $f(x) = 6x^4 + 22x^3 + 11x^2 - 80x - 40$. If $\frac{4}{3}$ is a zero of $f(x)$, then 4 is a factor of -40, and 3 is a factor of 6.

3. List all possible rational zeros of the following:

a. $f(x) = 4x^5 + x^4 - 2x^3 + 5x^2 + 8x + 16$

$$\frac{\pm 1, \pm 2, \pm 4, \pm 8\ \pm 16}{\pm 1, \pm 2, \pm 4} = \pm 1, \pm \tfrac{1}{2}, \pm \tfrac{1}{4}, \pm 2, \pm 4, \pm 8, \pm 16$$

4. Find all the zeros of the following functions:

a. $h(x) = 9x^4 + 5x^2 - 4$

possible zeros

$$\frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3, \pm 9}$$

$= \pm 1, \pm \frac{1}{3}, \pm \frac{1}{9},$
$\pm 2, \pm \frac{2}{3}, \pm \frac{2}{9}$
$\pm 4, \pm \frac{4}{3}, \pm \frac{4}{9}$

Try zeros until one works

$$\begin{array}{c|ccccc} \frac{2}{3} & 9 & 0 & 5 & 0 & -4 \\ & & 6 & 4 & 6 & 4 \\ \hline & 9 & 6 & 9 & 6 & 0 \end{array}$$

$h(x) = (3x-2)(9x^3 + 6x^2 + 9x + 6)$
$= (3x-2)[3x^2(3x+2) + 3(3x+2)]$
$= (3x-2)(3x^2+3)(3x+2)$
$= 3(3x-2)(x^2+1)(3x+2)$

$\boxed{\text{zeros: } x = \pm \frac{2}{3}, \pm i}$

b. $k(x) = 2x^4 - 5x^3 + 20x^2 - 45x + 18$

possible zeros:

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18}{\pm 1, \pm 2}$$

$$\begin{array}{c|ccccc} 2 & 2 & -5 & 20 & -45 & 18 \\ & & 4 & -2 & 36 & -18 \\ \hline & 2 & -1 & 18 & -9 & 0 \end{array}$$

$k(x) = (x-2)(2x^3 - x^2 + 18x - 9)$
$= (x-2)[x^2(2x-1) + 9(2x-1)]$
$= (x-2)(x^2+9)(2x-1)$

$\boxed{\text{zeros: } x = 2, 1/2, \pm 3i}$

c. $f(x) = 3x^3 - 2x^2 - 8x + 5$

possible zeros:

$$\frac{\pm 1, \pm 5}{\pm 1, \pm 3}$$

$$\begin{array}{c|cccc} \frac{5}{3} & 3 & -2 & -8 & 5 \\ & & 5 & 5 & -5 \\ \hline & 3 & 3 & -3 & 0 \end{array}$$

$f(x) = (3x-5)(3x^2+3x-3)$
$= 3(3x-5)(x^2+x-1)$

*quadratic formula

$$x = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$\boxed{\text{zeros: } x = 5/3, \frac{-1 \pm \sqrt{5}}{3}}$

KeyConcept Descartes' Rule of Signs

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function with real coefficients. Then

- the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this by an even number.

6 zeros

5. State the possible number of positive, real zeros, negative real zeros, and imaginary zeros of $f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$.

$f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$ $\quad$ $f(-x) = x^6 - 3x^5 - 4x^4 + 6x^3 + x^2 + 8x$

4 changes $\quad$ 2 changes

4 pos $\quad$ 2 neg
2 pos $\quad$ 2 neg $\quad$ 2 imaginary
0 pos $\quad$ 2 neg $\quad$ 4 imaginary

4 pos $\quad$ 0 neg $\quad$ 2 imaginary
2 pos $\quad$ 0 neg $\quad$ 4 imaginary
0 pos $\quad$ 0 neg $\quad$ 6 imaginary

6. State the possible number of positive, real zeros, negative real zeros, and imaginary zeros of $f(x) = 2x^5 + x^4 + 3x^3 - 4x^2 - x + 9$.

$f(x) = 2x^5 + x^4 + 3x^3 - 4x^2 - x + 9$ $\quad$ $f(-x) = -2x^5 + x^4 - 3x^3 - 4x^2 + x + 9$

2 changes $\quad$ 3 changes

5 zeros

2 pos $\quad$ 3 neg
0 pos $\quad$ 3 neg $\quad$ 2 imaginary
2 pos $\quad$ 1 neg $\quad$ 2 imaginary $\quad$ 0 pos $\quad$ 1 neg $\quad$ 4 imaginary

7. Write a polynomial function of least degree with integral coefficients, the zeros of which include -1 and $1 + 2i$.

*also $1-2i$

$f(x) = (x+1)(x-(1-2i))(x-(1+2i))$
$= (x+1)(x-1+2i)(x-1-2i)$
$= (x+1)[(x-1)^2 - 4i^2]$
$= (x+1)(x^2 - 2x + 1 + 4)$
$= (x+1)(x^2 - 2x + 5)$
$= x^3 - 2x^2 + 5x + x^2 - 2x + 5$
$= x^3 - x^2 + 3x + 5$

	x	-1	$2i$
x	x^2	$-x$	$\cancel{2ix}$
-1	$-x$	1	$\cancel{-2i}$
$-2i$	$\cancel{-2ix}$	$\cancel{2i}$	$-4i^2$

$x^2 - 2x + 1 - 4i^2$
$x^2 - 2x + 5$

8. Write a polynomial function of least degree with integral coefficients, the zeros of which include -3, 1, and $-3i$ and $3i$

$f(x) = (x+3)(x-1)(x-3i)(x+3i)$

$= (x^2+2x-3)(x^2-9i^2)$

$= (x^2+2x-3)(x^2+9)$

$= x^4 + 2x^3 - 3x^2 + 9x^2 + 18x - 27$

$= x^4 + 2x^3 + 6x^2 + 18x - 27$