

Recall:

1. Find the derivative for the following functions:

a.  $\sin x = y$

$$y' = \cos x$$

b.  $x^2 - 4 = y$

$$y' = 2x$$

What about  $\sin(x^2 - 4)$ ?

Finding Chain Rule:

1.  $y = 6x - 10$

$$y' = 6 \quad \rightarrow \quad y = 2(3x - 5)$$

composite function

$$y = 2u \quad \text{where } u = 3x - 5$$

$$\frac{dy}{du} = 2 \quad \leftarrow \text{"derivative of } y \text{ w/ respect to } u\text{"}$$

$$\frac{du}{dx} = 3 \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$6 = 2 \cdot 3$$

2.  $y = 9x^4 + 6x^2 + 1$

$$y = (3x^2 + 1)^2$$

composite of  $y = u^2$  where  $u = 3x^2 + 1$

$$\frac{dy}{du} = 2u \quad \frac{du}{dx} = 6x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x = 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x \end{aligned}$$

### RULE 8 The Chain Rule

If  $f$  is differentiable at the point  $u = g(x)$ , and  $g$  is differentiable at  $x$ , then the composite function  $(f \circ g)(x) = f(g(x))$  is differentiable at  $x$ , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \quad \star \text{ more common notation}$$

used above

In Leibniz notation, if  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where  $dy/du$  is evaluated at  $u = g(x)$ .

$\star$  derivative of outside, inside, derivative of inside

\* determine outside function and  
inside function

AB Calculus  
4.1 Chain Rule

1. Find the derivative of the following:

a.  $y = \sin(x^2 + x)$

$$f = \sin \square \quad g = x^2 + x$$

$$f' = \cos \square \quad g' = 2x$$

$$\begin{aligned}y' &= f'(g(x)) g'(x) \\&= \cos(x^2 + x)(2x) \\&= 2x \cos(x^2 + x)\end{aligned}$$

b.  $y = \cos(3x + \pi)$

$$f = \cos \square \quad g = 3x + \pi$$

$$f' = -\sin \square \quad g' = 3$$

$$\begin{aligned}y' &= -\sin(3x + \pi)(3) \\&= -3 \sin(3x + \pi)\end{aligned}$$

c.  $y = \tan(2x - x^3)$

$$f = \tan \square \quad g = 2x - x^3$$

$$f' = \sec^2 \square \quad g' = 2 - 3x^2$$

$$\begin{aligned}y' &= \sec^2(2x - x^3)(2 - 3x^2) \\&= 2 \sec^2(2x - x^3) - 3x^2 \sec^2(2x - x^3)\end{aligned}$$

$$y = 5 \cot(2x^{-1})$$

$$y' = 5(-\csc^2(2x^{-1}))(-2x^{-2}) \\ = \frac{10 \csc^2(2/x)}{x^2}$$

AB Calculus  
4.1 Chain Rule

d.  $y = 5 \cot\left(\frac{2}{x}\right)$

e.  $y = \sec(\tan x)$

$$y' = \underbrace{\sec(\tan x) \tan(\tan x) (\sec^2 x)}_{\text{derivative of outside ... inside}}$$

Recall:  $\frac{d}{dx}(\sec x) = \sec x \tan x$

f.  $y = (\csc x + \cot x)^{-1}$

$$y' = -(\csc x + \cot x)^{-2} (-\csc x \cot x - \csc^2 x) \\ = \frac{\csc x \cot x + \csc^2 x}{(\csc x + \cot x)^2} \\ = \frac{\csc x (\cot x + \csc x)}{(\csc x + \cot x)^2} \\ = \frac{\csc x}{\csc x + \cot x}$$

g.  $y = x^3(2x-5)^4$

$$y' = 3x^2(2x-5)^4 + x^3 \left[ 4(2x-5)^3(2) \right] \\ = 3x^2(2x-5)^4 + 8x^3(2x-5)^3$$

h.  $y = 4\sqrt{\sec x + \tan x}$

$$\begin{aligned}
 y' &= 4\left(\frac{1}{2}(\sec x + \tan x)^{-\frac{1}{2}}\right)(\sec x \tan x + \sec^2 x) \\
 &= \frac{2(\sec x \tan x + \sec^2 x)}{(\sec x + \tan x)^{\frac{1}{2}}} \\
 &= \frac{2 \sec x (\tan x + \sec x)}{(\sec x + \tan x)^{\frac{1}{2}}} \\
 &= 2 \sec x \sqrt{\sec x + \tan x}
 \end{aligned}$$

i.  $y = \frac{x}{\sqrt{1+x^2}} = \frac{x}{(1+x^2)^{\frac{1}{2}}} \quad * \text{ could change}$

to product rule if preferred

$$y' = \frac{(1+x^2)^{\frac{1}{2}}(1) - (x)\left(\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)\right)}{\left[(1+x^2)^{\frac{1}{2}}\right]^2} \text{ chain}$$

$$= \frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{(1+x^2)}$$

$$= \frac{(1+x^2)\left[(1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-\frac{3}{2}}\right]}{1+x^2}$$

j.  $y = (1+\cos 2x)^2$

$$y' = 2(1+\cos 2x)^1(-\sin 2x)(2)$$

$$= -4 \sin 2x(1+\cos 2x)$$

$$\begin{aligned}
 &= (1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-\frac{3}{2}} \\
 &= \frac{1}{\sqrt{1+x^2}} - \frac{x^2}{(1+x^2)^{\frac{3}{2}}} \\
 &= \frac{(1+x^2) - x^2}{(1+x^2)^{\frac{3}{2}}} \\
 &= \frac{1}{(1+x^2)^{\frac{3}{2}}}
 \end{aligned}$$

\* chain inside  
chain

$$y' = \sec 2x \tan 2x (2) \tan 2x + \sec(2x) \sec^2(2x) 2$$

AB Calculus  
4.1 Chain Rule

k.  $y = \sec(2x) \tan(2x)$

l.  $y = 2x\sqrt{\sec x}$

$$y' = 2\sqrt{\sec x} + 2x(\frac{1}{2}(\sec x)^{-\frac{1}{2}}(\sec x \tan x))$$

$$= 2\sqrt{\sec x} + \frac{x \sec x \tan x}{\sqrt{\sec x}}$$

$$= \frac{2 \sec x + x \sec x \tan x}{\sqrt{\sec x}}$$

$$= \frac{\sec x (2 + x \tan x)}{(\sec x)^{\frac{1}{2}}}$$

$$= \sqrt{\sec x} (2 + x \tan x)$$

m.  $y = 9 \tan\left(\frac{x}{3}\right)$

$$y' = 9 \sec^2\left(\frac{x}{3}\right) \left(\frac{1}{3}\right)$$

$$= 3 \sec^2\left(\frac{x}{3}\right)$$