Let f and g be inverse functions. What is true about inverses?

$$f(g(x)) = x$$

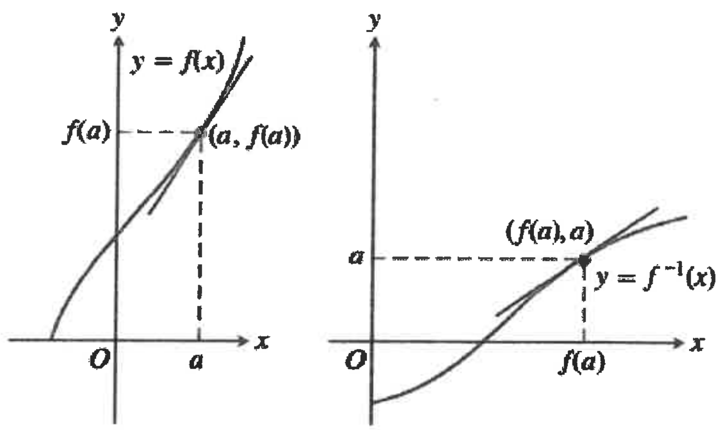
$$f'(g(x))\, g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

Theorem 1 **Derivatives of Inverse Functions**

If f is differentiable at every point of an interval I and $f'(x)$ is never zero on I, then f has an inverse g, and g is differentiable at every point of the interval $f(I)$. If $g(x) = x$, the **inverse function slope relationship** relates the derivative by the equation

$$g'(x) = \frac{1}{f'(g(x))}$$



The slopes are reciprocal: $\left.\dfrac{df^{-1}}{dx}\right|_{f(a)} = \dfrac{1}{\left.\dfrac{df}{dx}\right|_{a}}$

Ex) Let f and g be functionsthat are differentiable everywhere. If g is the inverse function of f and if $g(-2) = 5$ and $f'(5) = -\frac{1}{2}$, then $g'(-2) = ?$

$$g'(-2) = \frac{1}{f'(g(-2))}$$

$$= \frac{1}{f'(5)}$$

$$= \frac{1}{-\frac{1}{2}} = -2$$

Derivative Rules:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\cdot\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\cdot\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Examples:

1. Find the derivative of y with respect to the appropriate variable.

a. $\sin^{-1}(x) = y$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

b. $\cos^{-1}(\frac{1}{x}) = y$

$$\frac{1}{x} = x^{-1}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(1/x)^2}}(-x^{-2})$$

$$= \frac{1}{x^2\sqrt{1-(1/x)^2}}$$

$$= \frac{1}{x^2\sqrt{1-1/x^2}}$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{x^2\frac{\sqrt{x^2-1}}{\sqrt{x^2}}}$$

$$= \frac{1}{\frac{x^2\sqrt{x^2-1}}{x}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

c. $y = s\sqrt{1-s^2} + \cos^{-1}s$

$$\frac{dy}{ds} = \sqrt{1-s^2} + s\left(\tfrac{1}{2}(1-s^2)^{-1/2}(-2s)\right) + \left(-\frac{1}{\sqrt{1-s^2}}\right)$$

$$= \frac{\sqrt{1-s^2}}{1} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}}$$

$$= \frac{\sqrt{1-s^2}\sqrt{1-s^2} - s^2 - 1}{\sqrt{1-s^2}}$$

$$= \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$$

2. A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t)$. Find the velocity at the indicated value of t.

a. $x(t) = \sin^{-1}\left(\frac{\sqrt{t}}{4}\right),\ t = 4$

$\frac{1}{4}t^{1/2}$

$$x'(t) = \frac{1}{\sqrt{1-(\sqrt{t}/4)^2}} \cdot \frac{1}{8}t^{-1/2}$$

$$= \frac{1}{8t^{1/2}\sqrt{1-t/16}}$$

$$= \frac{1}{8(4)^{1/2}\sqrt{1-4/16}}$$

$$= \frac{1}{16\sqrt{12/16}}$$

$$= \frac{1}{16\,\sqrt{3}/2} = \frac{1}{8\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{24}$$

b. $x(t) = \tan^{-1}(t^2)$, $t = 1$

$$x'(t) = \frac{1}{1+(t^2)^2}(2t)$$

$$= \frac{2t}{1+t^4}$$

$$= \frac{2}{1+1^4}$$

$$= 1$$

4. **Find the derivative of y with respect to the appropriate variable.**

a. $y = \sec^{-1}(5s)$

$$\frac{dy}{dx} = \frac{1}{|5s|\sqrt{(5s)^2-1}} \cdot 5$$

$$= \frac{1}{5|s|\sqrt{25s^2-1}} \cdot 5$$

$$= \frac{1}{|s|\sqrt{25s^2-1}}$$

b. $y = \cot^{-1}\sqrt{t}$

$$y' = -\frac{1}{1+(\sqrt{t})^2} \quad \frac{1}{2}t^{-1/2}$$

$$= \frac{-1}{2t^{1/2}(1+t)}$$

$$= \frac{-1}{2\sqrt{t}(1+t)}$$

c. $y = \csc^{-1}\frac{x}{2}$

$$y' = -\frac{1}{|\frac{x}{2}|\sqrt{(x/2)^2-1}} \cdot \frac{1}{2}$$

$$= \frac{-1}{2 \cdot \frac{1}{2}|x|\sqrt{x^2/4-1}}$$

$$= \frac{-1}{|x|\sqrt{\frac{x^2}{4}-1}}$$

5. Let $f(x) = \cos x + 3x$
   a. Find $f(0)$ and $f'(0)$

$f(0) = 1$

$f'(x) = -\sin x + 3$

$f'(0) = 3$

   b. Find $f^{-1}(1)$ and $(f^{-1})'(1)$

original pt

$(1, 2)$ f^{-1} $(2, 1)$

$f^{-1}(1) = 0$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{3}$$