

4.3 Derivative of Inverse Trigonometric Functions

Let f and g be inverse functions. What is true about inverses?

$$f(g(x)) = x$$

* x and y swap

$$f'(g(x)) g'(x) = 1$$

$$f \rightarrow (x, y)$$

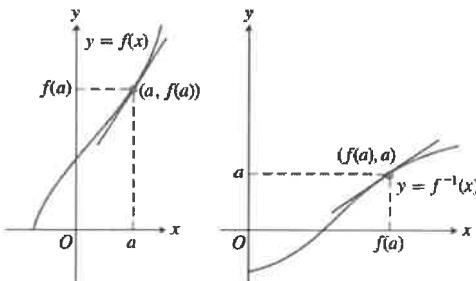
$$g'(x) = \frac{1}{f'(g(x))}$$

$$f^{-1} \rightarrow (y, x)$$

Theorem 1 Derivatives of Inverse Functions

If f is differentiable at every point of an interval I and $f'(x)$ is never zero on I , then f has an inverse g , and g is differentiable at every point of the interval $f(I)$. If $g(x) = x$, the **inverse function slope relationship** relates the derivative by the equation

$$g'(x) = \frac{1}{f'(g(x))}$$



$$\text{The slopes are reciprocal: } \frac{df^{-1}}{dx} \Big|_{f(a)} = \frac{1}{\frac{df}{dx}|_a}$$

1. Let f and g be function that are differentiable everywhere. If g is the inverse function of f and if $g(-2) = 5$ and $f'(5) = -\frac{1}{2}$, then $g'(-2) = ?$

$$\begin{aligned} g'(-2) &= \frac{1}{f'(g(-2))} \\ &= \frac{1}{f'(5)} \\ &= \frac{1}{-\frac{1}{2}} = -2 \end{aligned}$$

2. Let f be a differentiable function such that $f(3) = 15$, $f(6) = 3$, $f'(3) = -8$, and $f'(6) = -2$. The function g is differentiable and $g(x) = f^{-1}(x)$. What is the value of $g'(3)$?

- a. $-\frac{1}{2}$
- b. $-\frac{1}{4}$
- c. $\frac{1}{6}$
- d. $\frac{1}{3}$

* $g(3) \rightarrow (3, -)$

$$g^{-1} \rightarrow (-, 3)$$

- e. The value of $g'(3)$ cannot be determined with the given information.

$$\begin{aligned} g'(3) &= \frac{1}{f'(g(3))} \\ &= \frac{1}{f'(6)} = \frac{1}{-2} = -\frac{1}{2} \end{aligned}$$

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$$h'(x) = 5x^4 + 3 \quad h^{-1}(2) = ?$$

3. The function h is given by $h(x) = x^5 + 3x - 2$ and $h(1) = 2$. If h^{-1} is the inverse of h , what is the value of $(h^{-1})'(2)$?

- a. $\frac{1}{83}$
- b. $\frac{1}{8}$**
- c. $\frac{1}{2}$
- d. 1
- e. 8

$$(h^{-1})'(2) = \frac{1}{h'(h^{-1}(2))}$$

$$= \frac{1}{h'(1)}$$

$$= \frac{1}{8}$$

$$h'(1) = 5(1)^4 + 3 \\ = 8$$

4. Let $f(x) = (2x+1)^3$ and let $g(f(x)) = x$. Given that $f(0) = 1$, what is the value of $g'(1)$?

- a. $-\frac{2}{27}$
- b. $\frac{1}{54}$
- c. $\frac{1}{27}$
- d. $\frac{1}{6}$**
- e. 6

$$g'(1) = \frac{1}{f'(g(1))}$$

$$= \frac{1}{f'(0)}$$

$$= \frac{1}{6}$$

$$f'(x) = 3(2x+1)^2 \quad (2) \\ \rightarrow 6(2x+1)^2 \\ f'(0) = 6(1)^2 \\ = 6$$

- ~~5.~~ 5. If $f(x) = \sin x + 2x + 1$ and g is the inverse function of f , what is the value of $g'(1)$?

- a. $\frac{1}{3}$**
- b. 1
- c. 3
- d. $\frac{1}{2+\cos 1}$
- e. $2 + \cos 1$

$$g'(1) = \frac{1}{f'(g(1))}$$

$$= \frac{1}{f'(0)}$$

*when $f(x)=1 \quad g(1)$
 $f(x) = 1 = \sin x + 2x + 1$
 $1 = \sin x + 2x + 1$
 $0 = \sin x + 2x$
 $x=0$

$$f'(x) = \cos x + 2$$

$$f'(0) = \cos 0 + 2$$

$$= 3$$

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$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x| \cdot \sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x| \cdot \sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

1. Find the derivative of y with respect to the appropriate variable.

a. $y = \cos^{-1}\left(\frac{1}{x}\right)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(\frac{1}{x})^2}} (-x^{-2})$$

$$= \frac{1}{x^2 \sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{x^2 \sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{x^2 \frac{1}{x} \sqrt{x^2-1}}$$

$$= \boxed{\frac{1}{x \sqrt{x^2-1}}}$$

b. $y = s\sqrt{1-s^2} + \cos^{-1}s$

$$\frac{dy}{ds} = (1)\sqrt{1-s^2} + s\left(\frac{1}{2}(1-s^2)^{-1/2}(-2s)\right) + \left(-\frac{1}{\sqrt{1-s^2}}\right)$$

$$= \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}}$$

$$= \frac{1-s^2}{\sqrt{1-s^2}} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}}$$

$$= \frac{1-s^2-s^2-1}{\sqrt{1-s^2}}$$

$$= \boxed{\frac{-2s^2}{\sqrt{1-s^2}}}$$

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c. $x(t) = \tan^{-1}(t^2)$, $t = 1$

$$x'(t) = \frac{1}{1 + (t^2)^2} (2t)$$

$$= \frac{2t}{1+t^4}$$

$$x'(1) = \frac{2}{1+1}$$

$$= \boxed{1}$$

d. $y = \sec^{-1}(5s)$

$$y' = \frac{1}{|5s| \sqrt{(5s)^2 - 1}}$$

$$= \frac{5}{5|s| \sqrt{25s^2 - 1}}$$

$$= \boxed{\frac{1}{|s| \sqrt{25s^2 - 1}}}$$

e. $y = \cot^{-1}\sqrt{t}$

$$y' = -\frac{1}{1 + (\sqrt{t})^2} \left(\frac{1}{2}t^{-\frac{1}{2}} \right)$$

$$= -\frac{1}{2\sqrt{t}(1+t)}$$

f. $y = \csc^{-1}\frac{x}{2}$

$$y' = -\frac{1}{|\frac{x}{2}| \sqrt{(\frac{x}{2})^2 - 1}} \cdot \frac{1}{2}$$

$$= -\frac{1}{2(\frac{1}{2})|x| \sqrt{x^2/4 - 1}}$$

$$= -\frac{1}{|x| \sqrt{\frac{x^2 - 4}{4}}}$$

$$= -\frac{1}{|x| \sqrt{x^2 - 4}} \cdot \frac{1}{2}$$

$$= \boxed{-\frac{2}{|x| \sqrt{x^2 - 4}}}$$

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2. A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t)$. Find the velocity at the indicated value of t .

a. $x(t) = \sin^{-1}(\frac{\sqrt{t}}{4})$, $t = 4$

$$v(t) = \frac{1}{\sqrt{1 - (\frac{\sqrt{t}}{4})^2}} \cdot \frac{1}{8} t^{-\frac{1}{2}}$$

$$= \frac{1}{8\sqrt{t} \sqrt{1 - \frac{t}{16}}}$$

* plug in $t = 4$ here
or simplify further

$$= \frac{1}{8\sqrt{t} \sqrt{\frac{16-t}{16}}}$$

$$= \frac{1}{8\sqrt{t} \frac{\sqrt{16-t}}{4}}$$

$$= \frac{1}{2\sqrt{t} \sqrt{16-t}}$$

$$= \frac{1}{2\sqrt{16t - t^2}}$$

$$v'(4) = \frac{1}{2\sqrt{16(4) - 4^2}}$$

$$= \frac{1}{2\sqrt{48}}$$

$$= \frac{1}{8\sqrt{3}}$$

velocity @ $t = 4$ is

$$\frac{1}{8\sqrt{3}}$$

