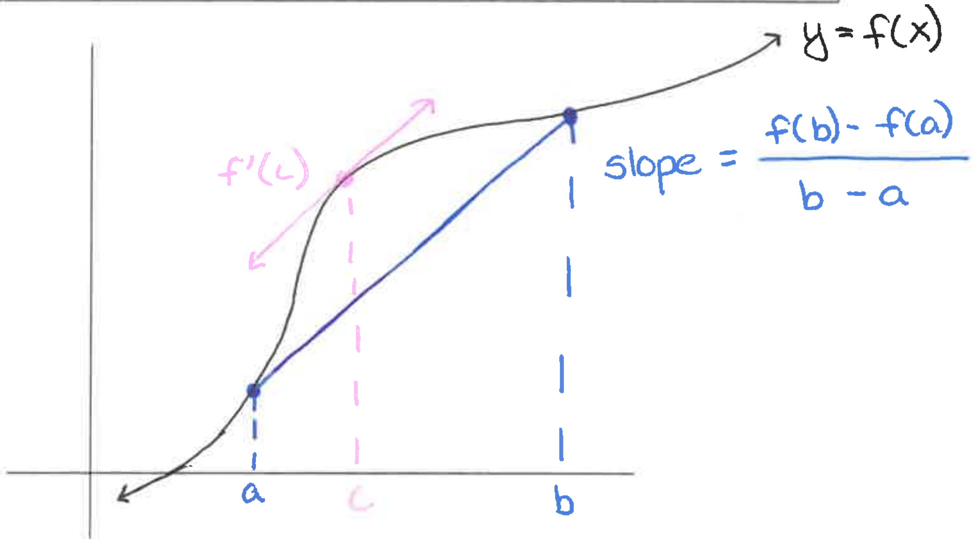
THEOREM 3 Mean Value Theorem for Derivatives

If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

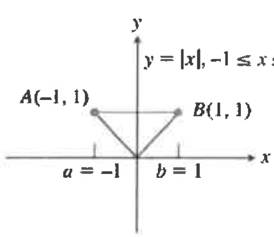
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Only says there is such a value, not where it is

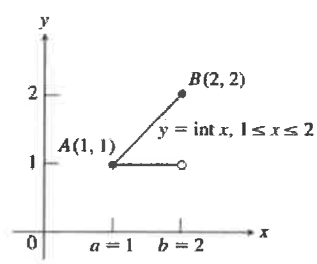
Conditions of Theorem cannot be relaxed!

$f(x) = |x|$



Not differentiable everywhere in (a, b)

$f(x) = \text{int}\, x$



Not continuous on closed interval $[a, b]$

1. Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval on the given interval. Then find each value of c in the interval (a, b) that satisfies the equation

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

a. $f(x) = x^{\frac{2}{3}}$ on $[0, 1]$

yes continuous on $[0,1]$

differentiable on $(0,1)$

*recall not differentiable at $x=0 \rightarrow$ cusp

$$f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{1-0}{1-0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$1 = \frac{2}{3}c^{-1/3}$$

$$\frac{3}{2} = c^{-1/3}$$

$$\boxed{\frac{8}{27} = c}$$

b. $f(x) = |x-1|$ on $[0, 4]$

no not differentiable at $x = 1$

↳ corner

c. $f(x) = \ln(x-1)$ on $[2, 4]$

yes continuous on $[2,4]$

differentiable on $(2,4)$

$$f'(c) = \frac{f(4)-f(2)}{4-2} = \frac{\ln(3)-\ln(1)}{2} = \frac{\ln 3}{2}$$

$$f'(x) = \frac{1}{x-1}$$

$$\frac{\ln 3}{2} = \frac{1}{c-1}$$

$$\frac{2}{\ln 3} + 1 = \boxed{c \approx 2.820}$$

d. $f(x) = \begin{cases} \sin^{-1} x, & -1 \le x < 1 \\ \frac{x}{2} + 1, & 1 \le x \le 3 \end{cases}$

continuous?

$$\lim_{x \to 1^-} f = \sin^{-1}(1) = \frac{\pi}{2}$$

$$\lim_{x \to 1^+} f = \frac{1}{2} + 1 = \frac{3}{2}$$

discontinuous at $x = 1$

so MVT doesn't apply

2. Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$.

a. $f(x) = \sqrt{x} + 1$

discontinuous from $[-1, 0)$

b. $f(x) = \begin{cases} x^3 + 3, & x < 1 \\ x^2 + 1, & x \geq 1 \end{cases}$

$\lim_{x \to 1^-} f = 4$ $\quad \lim_{x \to 1^+} f = 2$

discontinuous at $x = 1$

3. The interval $a \leq x \leq b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for the secant line AB and a tangent line to f in the interval (a, b) that is parallel to AB.

$f(x) = \sqrt{x-1}$ $\qquad 1 \leq x \leq 3$

$$m_{secant} = \frac{f(3) - f(1)}{3 - 1} = \frac{\sqrt{2} - 0}{2} = \frac{\sqrt{2}}{2}$$

*only need 1 pt could use 3 or any x on interval

$m = \frac{\sqrt{2}}{2}$ $\quad (1, 0)$

$$\boxed{y = \frac{\sqrt{2}}{2}(x - 1)}$$

$$f'(x) = \frac{1}{2}(x-1)^{-1/2} = \frac{1}{2\sqrt{x-1}}$$

$$\frac{\sqrt{2}}{2} = \frac{1}{2\sqrt{c-1}}$$

$$\sqrt{2} = \frac{1}{\sqrt{c-1}}$$

$$\left(\frac{1}{\sqrt{2}}\right)^2 = c - 1$$

$$\frac{1}{2} + 1 = c$$

$$\frac{3}{2} = c$$

$$f(3/2) = \sqrt{3/2 - 1} = \sqrt{1/2} = \frac{\sqrt{2}}{2}$$

$m = \frac{\sqrt{2}}{2}$ $\quad (3/2, \sqrt{2}/2)$

$$\boxed{y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(x - \frac{3}{2}\right)}$$

3. It took 20 sec for the temperature to rise from $0^{\circ}F$ to $212^{\circ}F$ when a thermometer was taken out of a freezer and placed in boiling water. Explain why at some point in the interval the mercury was rising at exactly $10.6^{\circ}F$.

$$f'(c) = \frac{f(20) - f(0)}{20}$$

$$= \frac{212 - 0}{20}$$

$$= 10.6$$

By MVT from time $[0, 20]$ there exists some $t \in [0, 20]$ such that the mercury is increasing by 10.6° F

4. A marathoner ran the New York City Marathon (26.2 mi) in 2.2h. Show that at least twice, the marathoner was running at exactly 11mph.

$$f'(c) = \frac{26.2}{2.2} = 11.909$$

the marathoners average velocity for the race was 11.909 mi/hr by the MVT there must exist a time $t \in [0, 2.2]$ where he passed 11 mph to get to the average velocity of 11.909 mph and a time $t \in [0, 2.2]$ where he passed 11 mph to finish the race and come to a complete stop.

DEFINITIONS Increasing Function, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f increases on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f decreases on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

COROLLARY 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.
2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

5. Use analytic methods to determine (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.

a. $g(x) = x^2 - x - 12$

a) $g'(x) = 2x - 1$

$0 = 2x - 1$

$\frac{1}{2} = x$

local min $(1/2, -49/4)$

b) $g'(x) = 2x - 1$

f': $-$ | $+$ at $\frac{1}{2}$

increasing from $(1/2, \infty)$ b/c $f'(x) > 0$

c) decreasing from $(-\infty, 1/2)$ b/c $f' < 0$

b. $k(x) = \frac{1}{x^2}$

$k'(x) = -2x^{-3}$

a) $0 = -2x^{-3}$

$0 = \frac{-2}{x^3}$ no x value so no local extrema

$k'(x)$ undefined at $x = 0$

f': $+$ | $-$ at 0

b) inc $(-\infty, 0)$ b/c $f' > 0$

c) dec $(0, \infty)$ b/c $f' < 0$

c. $f(x) = e^{-0.5x}$

$f'(x) = -0.5e^{-0.5x}$

a) $0 = -0.5e^{-0.5x}$

no local extrema

never undefined

b) never inc b/c $f'(x) \not> 0$

c) $(-\infty, \infty)$ b/c $f'(x) < 0$

d. $g(x) = x^{\frac{1}{3}}(x+8)$

$g'(x) = \frac{1}{3}x^{-2/3}(x+8) + x^{1/3}$

$= \dfrac{x+8}{3x^{2/3}} + x^{1/3}$

$= \dfrac{x+8 + 3x^{2/3}x^{1/3}}{3x^{2/3}}$

$= \dfrac{x+8+3x}{3x^{2/3}}$

$= \dfrac{4x+8}{3x^{2/3}}$

a) $0 = \dfrac{4x+8}{3x^{2/3}}$

$x = -2$

local min at $(-2, -7.56)$

$g'(x)$ undefined at $x=0$

f': $-$ | -2 | $+$ | 0 | $+$

inc $(-2, \infty)$ b/c $f'>0$

dec $(-\infty, -2)$ b/c $f'<0$

e. $k(x) = \frac{x}{x^2-4}$

$k'(x) = \dfrac{(x^2-4)(1) - x(2x)}{(x^2-4)^2}$

$= \dfrac{x^2-4-2x^2}{(x^2-4)^2}$

$= \dfrac{-x^2-4}{(x^2-4)^2}$

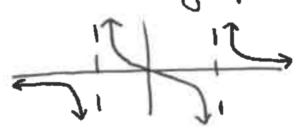
$0 = \dfrac{-x^2-4}{(x^2-4)^2}$

no extrema

$k'(x)$ und at $x = \pm 2$

$-$ | -2 | $-$ | 2 | $-$

* unusual check
$k(x)$ graph



dec everywhere
$(-\infty, \infty)$

f. $g(x) = 2x + \cos x$

$g'(x) = 2 - \sin x$

$0 = 2 - \sin x$

$2 = \sin x$

no extrema

g' never undefined

* $\sin x$ bounded between -1 and 1

$\Rightarrow g'(x)$ always >0 (pos)

g increasing $(-\infty, \infty)$ b/c $g'(x) > 0$

COROLLARY 2 Functions with $f' = 0$ are Constant

If $f'(x) = 0$ at each point of an interval I, then there is a constant C for which $f(x) = C$ for all x in I.

#Don't forget your cupcake

COROLLARY 3 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constant C such that $f(x) = g(x) + C$ for all x in I.

DEFINITION Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation.**

6. Find all possible functions f with the given derivative:

a. $f'(x) = \sin x$

$f(x) = -\cos x + C$

b. $f'(x) = \frac{1}{x-1},\ x > 1$

$f(x) = \ln|x-1| + C$

$= \ln(x-1) + C$

7. Find the function with the given derivative whose graph passes through the point P.

a. $f'(x) = \frac{1}{4x^{\frac{3}{4}}}$ $P(1, -2)$

$f'(x) = \frac{1}{4} x^{-3/4}$

$f(x) = x^{1/4} + C$

$f(1) = -2 = 1^{1/4} + C$

$-2 = 1 + C$

$-3 = C$

$f(x) = x^{1/4} - 3$

b. $f'(x) = 2x + 1 - \cos x$ $P(0, 3)$

$f(x) = x^2 + x - \sin x + C$

$3 = 0^2 + 0 - \sin 0 + C$

$3 = C$

$f(x) = x^2 + x - \sin x + 3$