THEOREM 3 Mean Value Theorem for Derivatives

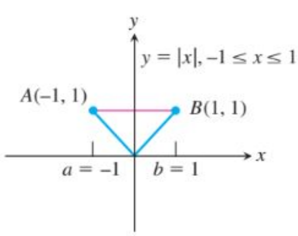
If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

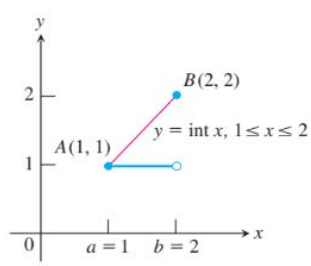
Only says there is such a value, not where it is

Conditions of Theorem cannot be relaxed!

$f(x) = |x|$



$f(x) = \text{int}\, x$



1. Show that the function $f(x) = x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval on the given interval. Then find each value of c in the interval (a, b) that satisfies the equation

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

a. $f(x) = x^{\frac{2}{3}}$ on $[0, 1]$

c. $f(x) = \ln(x-1)$ on $[2, 4]$

b. $f(x) = |x-1|$ on $[0, 4]$

d. $$f(x) = \begin{cases} \sin^{-1} x, & -1 \leq x < 1 \\ \frac{x}{2} + 1, & 1 \leq x \leq 3 \end{cases}$$

2. Explain why each of the following functions fails to satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$.

a. $f(x) = \sqrt{x} + 1$

b. $\begin{cases} x^3 + 3, & x < 1 \\ x^2 + 1, & x \geq 1 \end{cases}$

3. The interval $a \leq x \leq b$ is given. Let $A = (a, f(a))$ and $B = (b, f(b))$. Write an equation for the secant line AB and a tangent line to f in the interval (a, b) that is parallel to AB.

$f(x) = \sqrt{x-1}$ $\qquad 1 \leq x \leq 3$

3. It took 20 sec for the temperature to rise from $0^{\circ}F$ to $212^{\circ}F$ when a thermometer was taken out of a freezer and placed in boiling water. Explain why at some point in the interval the mercury was rising at exactly $10.6^{\circ}F$.

4. A marathoner ran the New York City Marathon in 2.2h. Show that at least twice, the marathoner was running at exactly 11mph.

DEFINITIONS Increasing Function, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

COROLLARY 1 Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.
2. If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

5. Use analytic methods to determine (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.

a. $g(x) = x^2 - x - 12$

b. $k(x) = \frac{1}{x^2}$

c. $f(x) = e^{-0.5x}$

d. $g(x) = x^{\frac{1}{3}}(x + 8)$

e. $k(x) = \frac{x}{x^2 - 4}$

f. $g(x) = 2x + \cos x$

COROLLARY 2 Functions with $f' = 0$ are Constant

If $f'(x) = 0$ at each point of an interval I, then there is a constant C for which $f(x) = C$ for all x in I.

COROLLARY 3 Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constant C such that $f(x) = g(x) + C$ for all x in I.

DEFINITION Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation.**

6. Find all possible functions f with the given derivative:

a. $f'(x) = \sin x$

b. $f'(x) = \frac{1}{x-1}$, $x > 1$

7. Find the function with the given derivative whose graph passes through the point P.

a. $f'(x) = \frac{1}{4x^{\frac{3}{4}}}$ $P(1, -2)$

b. $f'(x) = 2x + 1 - \cos x$ $P(0, 3)$