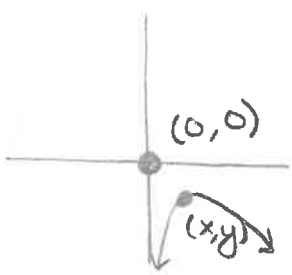
Steps:

1. What are you trying to optimize
2. Write all equations related to your problem.
   a. Objective: equation you are maximizing/minimizing
   b. Constraint: what limits your equation in objective equation
      i. There are times where there is no constraint equation or multiple constraint equations.
3. Substitute the constraint into the objective
4. Find the first derivative.
5. Set the derivative equal to zero.
6. Solve to find the critical values.
7. Determine which critical values answer the question being asked.

Examples:

1. Given the curve $y = \ln x - x$. Find the coordinates of the point on the curve that is closest to the origin.



$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$D = \sqrt{(x-0)^2 + (y-0)^2}$ * (0,0)

$D = \sqrt{x^2 + y^2}$ * 1 variable $y = \ln x - x$

$D = \sqrt{x^2 + (\ln x - x)^2}$

small → $\sqrt{}$ only makes smaller

$M = x^2 + (\ln x - x)^2$

$M' = 2x + 2(\ln x - x)(1/x - 1)$

$0 = 2x + 2(\ln x - x)(1/x - 1)$

$x \approx 0.632784$

$y \approx -1.09041$

2. The sum of two non-negative numbers is 30. Find the largest possible product that results when the square of one of the numbers is multiplied by the second number.

constraint → $x + y = 30$

$y = 30 - x$

$P = x^2 y$ ← objective

maximizing product

$P = x^2(30 - x)$

$P = 30x^2 - x^3$

$P' = 60x - 3x^2$

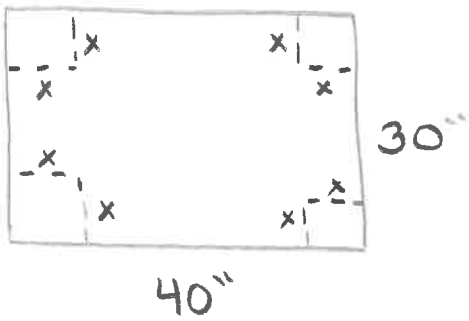
$0 = 3x(20 - x)$

$x = 0, 20$

$x = 20$
$y = 10$

max = $P(20) = 20^2(10)$
$= 4{,}000$

3. A 30" x 40" rectangular sheet of cardboard is made into a box without a top by cutting squares of equal size out of each corner and folding up the sides. Find its maximum volume and the corresponding dimensions of the box.



$V = lwh$

$= (30-2x)(40-2x)(x)$

$= (30-2x)(40x - 2x^2)$

$= 1200x - 80x^2 - 60x^2 + 4x^3$

$= 4x^3 - 140x^2 + 1200x$

$V' = 12x^2 - 280x + 1200$

$0 = 12x^2 - 280x + 1200$

$x = 5.657$

$V = 3032.303$

$5.657 \times 28.685 \times 18.685$

4. Farmer Alfalfa will use 600m of fencing to build a corral in the shape of a semicircle on top of a rectangle. Find the dimensions that would maximize the area of the corral.

$X = 2r$

$A = xy + \frac{1}{2}\pi r^2$

$A = 2ry + \frac{1}{2}\pi r^2$

objective

$P = x + 2y + \frac{1}{2}(2\pi r)$

$P = 2r + 2y + \pi r$

$600 = 2r + 2y + \pi r$

constraint

$2y = 600 - 2r - \pi r$

$y = 300 - r - \frac{\pi}{2} r$

$A = 2r(300 - r - \frac{\pi}{2} r) + \frac{1}{2}\pi r^2$

$A = 600r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$

$= 600r - 2r^2 - \frac{\pi}{2} r^2$

$A' = 600 - 4r - \pi r$

$= 600 - r(4 + \pi)$

$0 = 600 - r(4 + \pi)$

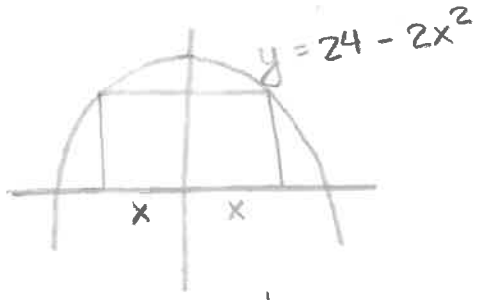
$\frac{600}{4+\pi} = r \approx 84.015$

$r \approx 84.015$

$x \approx 168.029$

$y \approx 84.015$

5. A rectangle has its base on the x-axis and its two upper corners on the parabola $y = 24 - 2x^2$. What is the largest possible area of the rectangle?



* know same x value b/c parabola wasn't shifted left/ right

$A = 2xy$ objective

$y = 24 - 2x^2$ constraint

$A = 2x(24 - 2x^2)$

$A = 48x - 4x^3$

$A' = 48 - 12x^2$

$0 = 48 - 12x^2$

$\pm 2 = x$

$A = 2(2)(24 - 2(2)^2) = 64$

6. Angus is on one side of a river that is 50m wide. He wants to reach a point 200m downstream on the opposite side as quickly as possible. He plans to swim diagonally across the river, then run the rest of the way. Angus can swim at 1.5m/s and run at 4m/s. Where should he come ashore?

$v = \frac{d}{t}$

$t = \frac{d}{v}$

$t_{swim} = \frac{\sqrt{50^2 + x^2}}{1.5}$ $\quad t_{run} = \frac{200 - x}{4}$

$t_{total} = \frac{\sqrt{50^2 + x^2}}{1.5} + \frac{200 - x}{4}$

$t' = \frac{\frac{1}{2}(50^2 + x^2)^{-1/2}(2x)}{1.5} + -\frac{1}{4}$

$0 = \frac{x}{1.5(50^2 + x^2)^{1/2}} - \frac{1}{4}$

$x \approx 20.226$

20.226 m downstream then run 179.774 m

7. If $xy^2 = 128$ and $z = x^2 + y^2$, what is the minimum value of z?

$xy^2 = 128$

$y^2 = \frac{128}{x}$

$z = x^2 + (128/x)$

$z' = 2x + (-128x^{-2})$

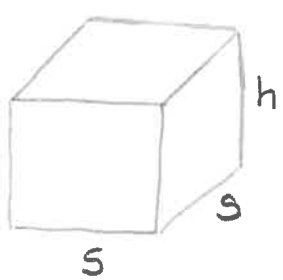
$0 = 2x - \frac{128}{x^2}$

$x = 4$ $\quad y^2 = \frac{128}{4} = 32$

$z = 4^2 + 32$

$= 48$

8. An open top box with a square bottom and rectangular sides has a volume of 72 m^3. It is constructed out of two different materials. The cost of the bottoms is $\$40/m^2$ and the cost of the sies is $\underbrace{\$30/m^2}_{4 \text{ sides}}$. Find the dimensions of the box that minimize the total cost.



$V = 72 = s(s)(h)$

$72 = s^2 h$

$h = \frac{72}{s^2}$

cost minimizing

$C = 40s^2 + 4[30hs]$

$= 40s^2 + 120hs$

$= 40s^2 + 120\left(\frac{72}{s^2}\right)s$

$= 40s^2 + 8640s^{-1}$

$C' = 80s - 8640s^{-2}$

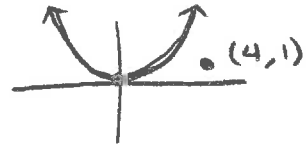
$0 = 80s - 8640s^{-2}$

$s = 4.762$

$h = 3.175$

$4.762 \times 4.762 \times 3.175$

$y = \frac{1}{2}x^2$



9. The point on the curve $2y = x^2$ nearest to $(4, 1)$ is...

a. $(0,0)$ b. $(2,2)$ c. $(\sqrt{2}, 1)$ d. $(2\sqrt{2}, 4)$ e. $(4, 8)$

$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(x-4)^2 + (\frac{1}{2}x^2 - 1)^2}$

minimize

$M = (x-4)^2 + (\frac{1}{2}x^2 - 1)^2$

$M' = 2(x-4) + 2(\frac{1}{2}x^2 - 1)(x)$

$= 2x - 8 + x^3 - 2x$

$M' = x^3 - 8$

$0 = x^3 - 8$

$\sqrt[3]{8} = x$

$2 = x$

$y = \frac{1}{2}(2)^2$

$= 2$

$(2, 2)$

10. If $y = 2x - 8$, what is the minimum value of the product xy?

constraint objective

a. -16 b. -8 c. -4 d. 0 e. 2

$P = xy$

$= x(2x - 8)$

$= 2x^2 - 8x$

$P' = 4x - 8$

$0 = 4x - 8$

$2 = x$

$y = 2(2) - 8$

$= -4$

$P = (2)(-4)$

$= -8$

11. If $3p + 2q = 600$, the maximum value of pq is ...

a. 100 b. 150 c. 600 d. 15,000 e. 60,000

pq

$= p(300 - 3/2\, p)$

$= 300p - 3/2\, p^2$

$\frac{d}{dx}(300p - 3/2\, p^2)$

$= 300 - 3p$

$0 = 300 - 3p$

$3p + 2q = 600$

$2q = 600 - 3p$

$q = 300 - 3/2\, p$

$p = 100$

$q = 300 - 3/2(100)$

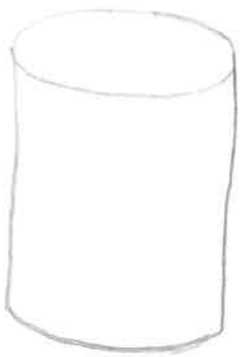
$= 150$

$pq = 100(150)$

$= 15,000$

12. Consider all right circular cylinders for which the sum of the height and circumference is 30 centimeters. What is the radius of the one with maximum volume?

a. 3 cm b. 10 cm c. 20 cm d. $\frac{30}{\pi^2}$ cm e. $\frac{10}{\pi}$ cm



$h + C = 30$

$h + 2\pi r = 30$

$h = 30 - 2\pi r$

$V = \pi r^2 h$

$V = \pi r^2(30 - 2\pi r)$

$V = 30\pi r^2 - 2\pi^2 r^3$

$V' = 60\pi - 6\pi^2 r^2$

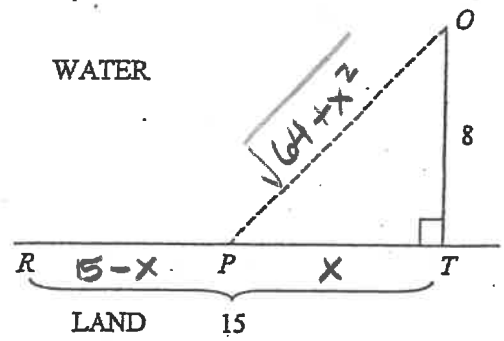
$0 = 60\pi - 6\pi^2 r^2$

$0 = -6\pi r(-10 + \pi r)$

$r = 0,\ 10/\pi$

$r = 10/\pi$

(3)



An oil well at O is in the ocean, 8 miles from T on a straight shoreline. $RT = 15$ miles. The oil has to go from O to R. The cost of laying pipe per mile is \$90,000 underwater and \$54,000 on land. The cheapest method of placing the pipe is to lay $\overline{OP}$ underwater and $\overline{PR}$ on land, where P is some point on $\overline{RT}$. What is the amount of money that can be saved by using this method instead of going directly from O to R underwater?

min cost

$C_w = 90{,}000$

$C_l = 54{,}000$

$$C = C_w + C_l$$

$$= 90{,}000\left(\sqrt{64+x^2}\right) + 54{,}000(15-x)$$

$$= 90{,}000\sqrt{64+x^2} + 810{,}000 - 54{,}000x$$

$$C' = 45{,}000(64+x^2)^{-1/2}(2x) - 54{,}000$$

$$0 = \frac{90{,}000\,x}{\sqrt{64+x^2}} - 54{,}000$$

$$x = 6$$

$$C = 90{,}000\sqrt{64+6^2} + 54{,}000(15-6)$$

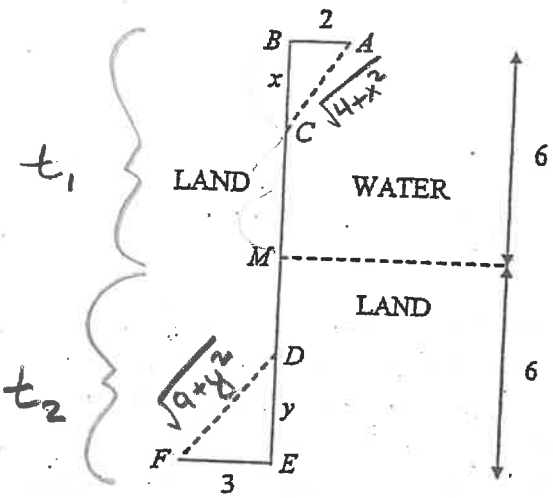
$$C = \$1{,}386{,}000$$

$$C_{water\ \overline{OR}} = 90{,}000\sqrt{64+15^2}$$

$$= \$1{,}530{,}000$$

$$\boxed{\text{savings} = \$144{,}000}$$

(4)



6mph swimming
10 mph cycling
8 mph running

In a triathlon event, Janet must swim from point A (2 miles from B) to any point C on $\overline{BM}$, then bicycle to any point D on $\overline{ME}$ and finally run to point F (3 miles from E). $BM = ME = 6$ miles. Janet (with the aid of a strong current) can average 6 mph swimming, 10 mph cycling, and 8 mph running. Janet figured out point C so as to go from A to C to M in the shortest possible time. She also figured out point D to minimize the time from M to D to F. How far did she cycle from C to D?

$$v = \frac{d}{t} \qquad t = \frac{d}{v}$$

$$t_1 = \frac{\sqrt{4+x^2}}{6} + \frac{6-x}{10} \qquad t_2 = \frac{\sqrt{9+y^2}}{8} + \frac{6-y}{10}$$

$$t_1' = \frac{x}{6\sqrt{x^2+4}} - \frac{1}{10} = 0 \qquad t' = \frac{y}{8\sqrt{9+y^2}} - \frac{1}{10} = 0$$

$$x = \frac{3}{2} \qquad y = 4$$

$$\text{cycles} = 6 - x + 6 - y$$

$$= 6 - 3/2 + 6 - 4$$

$$= 13/2 = 6.5$$