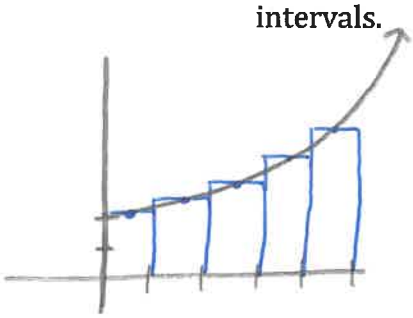
2018-19
BC Calculus
6.1 Estimating with Finite Sums
6.2 Definite Integrals
6.3 Definite Integrals and Antiderivative

Riemann Sums

- RRAM (Right Rectangular Approximation Method)
- LRAM (Left Rectangular Approximation Method)
- MRAM (Midpoint Rectangular Approximation Method)

1. A particle starts at $x = 0$ and moves along the x-axis with velocity $v(t) = t^2 + 2$ for time $t \geq 0$. Where is the particle at $t = 5$? Approximate the area under the curve using five rectangles of equal width and heights determined by the midpoints of the intervals.



$[0,1]\ [1,2]\ [2,3]\ [3,4]\ [4,5]$

$1/2 \quad 3/2 \quad 5/2 \quad 7/2 \quad 9/2$

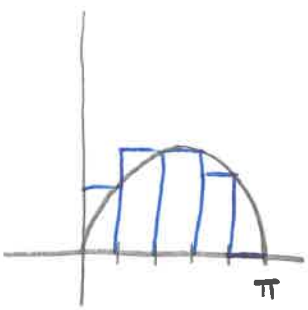
$$MRAM = 1(v(1/2)) + 1(v(3/2)) + 1(v(5/2)) + 1(v(7/2)) + 1(v(9/2))$$

$$= \frac{9}{4} + \frac{17}{4} + \frac{33}{4} + \frac{57}{4} + \frac{89}{4}$$

$$\boxed{= 51.25}$$

2. Use RRAM with $n = 5$ to estimate the area of the region enclosed between the graph of f and the x-axis for $a \leq x \leq b$

$f(x) = \sin x,\ a = 0,\ b = \pi$



$\frac{\pi - 0}{5} = \frac{\pi}{5}$ subintervals

$$RRAM = \frac{\pi}{5}(\sin(\pi/5)) + \frac{\pi}{5}(\sin(2\pi/5)) + \ldots$$

or

$$= \frac{\pi}{5}\left(\sin(\pi/5) + \sin(2\pi/5) + \sin(3\pi/5) + \sin(4\pi/5) + \sin\pi\right)$$

$$\boxed{= 1.934}$$

3.

Distance Traveled The table below shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine, using 10 subintervals of length 1 with (a) left-endpoint values (LRAM) and (b) right-endpoint values (RRAM).

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

a) LRAM = $1(0+12+22+10+5+13+11+6+2+6) = 87$ in

b) RRAM = $1(12+22+10+5+13+11+6+2+6+0) = 87$ in

THEOREM 1 The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

The Definite Integral of a Continuous Function on $[a, b]$

Let f be continuous on $[a, b]$, and let $[a, b]$ be partitioned into n subintervals of equal length $\Delta x = (b - a)/n$. Then the definite integral of f over $[a, b]$ is given by

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x,$$

where each c_k is chosen arbitrarily in the k^{th} subinterval.

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$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)\,dx.$$

Upper limit of integration

Integral sign

Lower limit of integration

The function is the **integrand.**

x is the **variable of integration.**

$$\int_a^b f(x)\,dx$$

Integral of f from a to b

When you find the value of the integral, you have **evaluated the integral.**

DEFINITION Area Under a Curve (as a Definite Integral)

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ from a to b** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

$$\text{Area} = -\int_a^b f(x)\,dx \quad \text{when} \quad f(x) \le 0.$$

$$\int_a^b f(x)\,dx = (\text{area above the } x\text{-axis}) - (\text{area below the } x\text{-axis}).$$

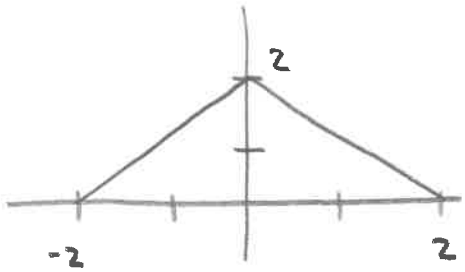
THEOREM 2 The Integral of a Constant

If $f(x) = c$, where c is a constant, on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b - a).$$

5. Use the graph of the integrand and area to evaluate the integral:

$$\int_{-2}^{2} (2 - |x|)\, dx$$



$= \frac{1}{2}(4)(2)$

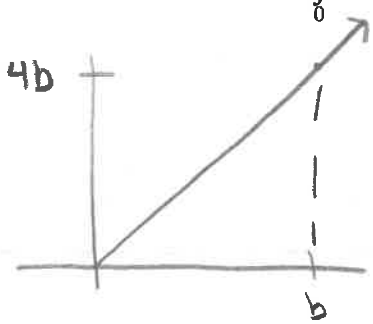
$\boxed{= 4}$

triangle

$A = \frac{1}{2}bh$

6. Use areas to evaluate the integral:

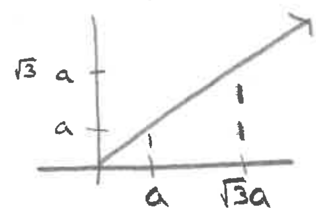
a. $\int_{0}^{b} 4x\, dx, \quad b > 0$



$\int_0^b 4x\,dx = \frac{1}{2}(b)(4b)$

$\boxed{= 2b^2}$

b. $\int_{a}^{\sqrt{3}a} x\, dx, \quad a > 0$



$A = \frac{1}{2}h(b_1 + b_2)$

$h = \sqrt{3}a - a$

$b_1 = a$

$b_2 = \sqrt{3}a$

$\int_a^{\sqrt{3}a} = \frac{1}{2}(\sqrt{3}a - a)(a + \sqrt{3}a)$

$= \frac{1}{2}(\sqrt{3}a^2 + 3a^2 - a^2 - \sqrt{3}a^2)$

$= \boxed{a^2}$

7. Find the points of discontinuity of the integrand on the interval of integration, and use area to evaluate the integral.

$$\int_{-5}^{6} \frac{9-x^2}{x-3}\, dx$$

$\frac{9-x^2}{x-3} = \frac{-(x^2-9)}{x-3}$

$= \frac{-(x-3)(x+3)}{x-3}$

$= -(x+3) \quad x \neq 3$

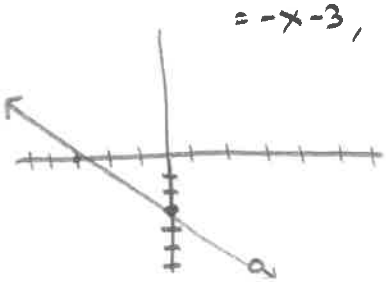
$= -x-3, \quad x \neq 3$

discontinuous at $x = 3$

$\int_{-5}^{6} \frac{9-x^2}{x-3}\,dx = \frac{1}{2}(2)(2) + \frac{1}{2}(9)(-9)$

$= 2 - 81/2$

$\boxed{= -\frac{77}{2}}$



*2 triangles

Table 5.3 Rules for Definite Integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero:* $\int_a^a f(x)\,dx = 0$ — Also a definition

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$$f(x) \ge 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0 \qquad g = 0$$

9. Suppose that f and g are continuous functions and that

$$\int_1^2 f(x)dx = -3, \quad \int_1^5 f(x)dx = 8, \quad \int_1^5 g(x)dx = 10$$

Find each integral below:

a. $\int_2^2 g(x)dx = \boxed{0}$

b. $\int_5^1 g(x)dx = \boxed{-8}$

c. $\int_1^2 3f(x)dx = 3(-3) = \boxed{-9}$

d. $\int_2^5 f(x)dx = \int_1^5 f(x)dx - \int_1^2 f(x)\,dx$

$= 8 - (-3) = \boxed{11}$

e. $\int_1^5 [f(x) - g(x)]dx = 8 - 10$

$\boxed{= -2}$

f. $\int_1^5 [4f(x) - g(x)]dx = 4(8) - 10$

$= \boxed{22}$

10. Suppose that h is continuous and that

$$\int_{-1}^{1} h(r)dr = 0 \quad \text{and} \quad \int_{-1}^{3} h(r)dr = 5$$

Find each integral.

*u dummy variable

a. $\int_1^3 h(r)dr = \int_{-1}^{3} h(r)dr - \int_{-1}^{1} h(r)dr$

$= 5$

b. $-\int_3^1 h(u)du = \int_1^3 h(u)du$

$= 5$

11. Interpret the integrand as the rate of change of a quantity and evaluate the integral using the antiderivative of the quantity:

a. $\int_0^{\frac{\pi}{2}} \cos x\, dx$

$= \sin x \Big|_0^{\pi/2}$

$= \sin \pi/2 - \sin 0$

$= 1$

b. $\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$

$= \tan x \Big|_0^{\pi/4}$

$= \tan \pi/4 - \tan 0$

$= 1$

c. $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx$

$= \sin^{-1} x \Big|_0^{1/2}$

$= \sin^{-1}(1/2) - \sin^{-1}(0)$

$= \pi/6$

DEFINITION Average (Mean) Value

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

12. Find the average value of the function on the interval, using antiderivatives to compute the integral.

a. $y = \frac{1}{x}$, $[e, 2e]$

$= \frac{1}{2e-e} \int_e^{2e} \frac{1}{x} dx$

$= \frac{1}{e} \ln|x| \Big|_e^{2e}$

$= \frac{1}{e}(\ln 2e - \ln e)$

$= \frac{1}{e} \ln \frac{2e}{e} = \boxed{\frac{1}{e} \ln 2}$

b. $y = \frac{1}{1+x^2}$, $[0, 1]$

$= \frac{1}{1-0} \int_0^1 \frac{1}{1+x^2} dx$

$= \tan^{-1} x \Big|_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \boxed{\frac{\pi}{4}}$

c. $y = \sec x \tan x$, $[0, \frac{\pi}{3}]$

$= \frac{1}{\pi/3 - 0} \int_0^{\pi/3} \sec x \tan x\, dx$

$= \frac{3}{\pi} \sec x \Big|_0^{\pi/3}$

$= \frac{3}{\pi}(\sec \pi/3 - \sec 0)$

$= \frac{3}{\pi}(2-1) = \boxed{\frac{3}{\pi}}$

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Most Difficult First:

Pg. 278: #19
Pg. 292: #52, 54
Pg. 300: #40