DEFINITION Differential Equation

An equation involving a derivative is called a **differential equation. The order of a differential equation** is the order of the highest derivative involved in the equation.

1. Find the general solution of the exact differential equation:

   a. $\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2}\ (x > 0)$

   $y = \ln|x| + \frac{1}{x} + C$

   b. $\frac{dy}{dx} = \sec^2 x + 2x + 5$

   $y = \tan x + x^2 + 5x + C$

2. Find the particular solution to the equation $\frac{dy}{dx} = e^x - 6x^2$ whose graph passes through the point $(1, 0)$.

   $y = e^x - 2x^3 + C$

   $0 = e^1 - 2(1)^3 + C$

   $2 - e = C$

   $y = e^x - 2x^3 + 2 - e$

3. Find the particular solution to the differential equation $\frac{dy}{dx} = 2e^x - \cos x$ and $y = 3$ when $x = 0$.

   $y = 2e^x - \sin x + C$

   $3 = 2e^0 - \sin 0 + C$

   $3 = 2 + C$

   $1 = C$

   $y = 2e^x - \sin x + 1$

4. Find $f(-2)$ with the differential equation $f'(x) = e^{-x^2}$ and $f(7) = 3$.

$$\int_{-2}^{7} e^{-x^2}\,dx = f(7) - f(-2)$$

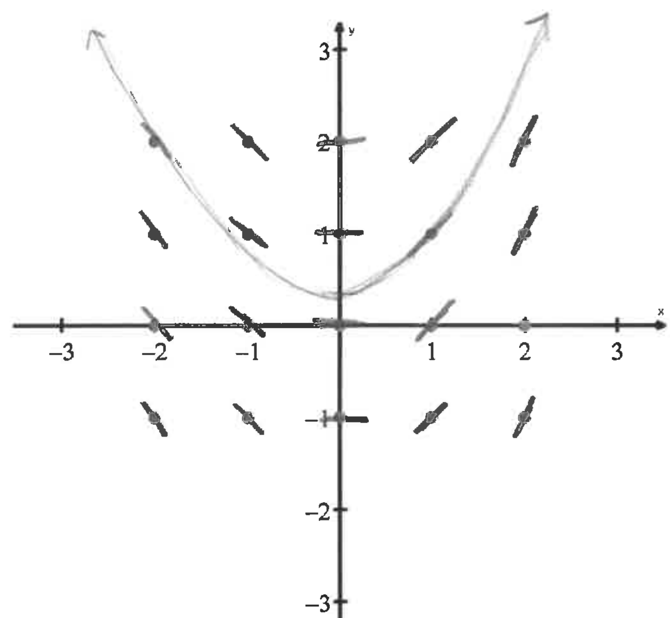
$$\int_{-2}^{7} e^{-x^2}\,dx = 3 - f(-2)$$

$$f(-2) = 1.232$$

Not all differential equations can be solved using separation of variables. These are called inseparable differential equations. You can graphically represent the solution to a differential equation using a **slope field**. Slope fields use short line segments to represent the tangent lines of the solution to the differential equation at many different points. Together, the many segments give a fuller picture of what the solution would look like.

5. Use separation of variables to find the solution to the differential equation $\frac{dy}{dx} = x$. Then, graph the slope field for the differential equation at the points indicated on the graph.

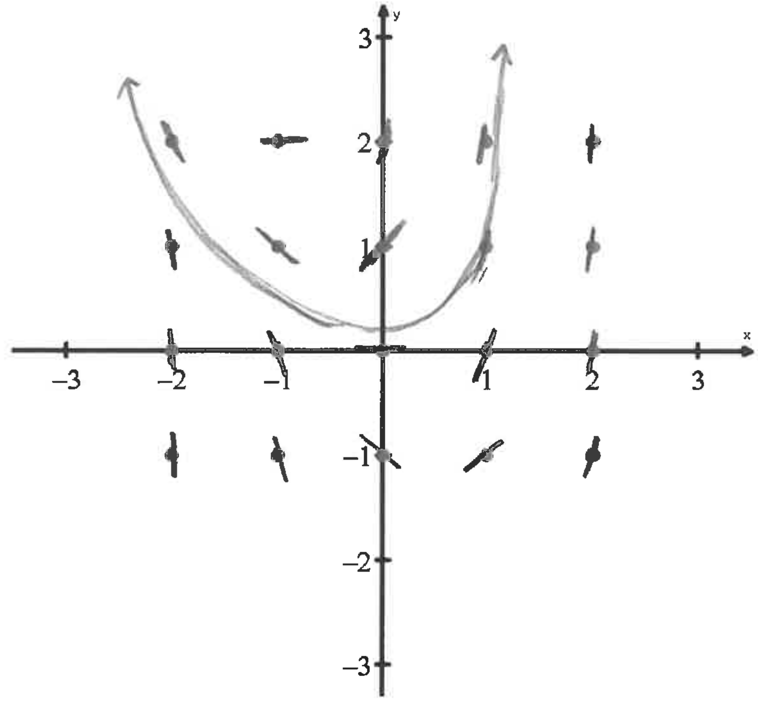
sketch particular solution if initial condition is (1, 1)



You can see the various solutions for the general solution to $\frac{dy}{dx} = x$ based on the value of C. You can also graph a particular solution on a slope field if you are given a point that the curve passes through. Begin at the initial condition and use the slope to sketch an approximate path for the solution curve.

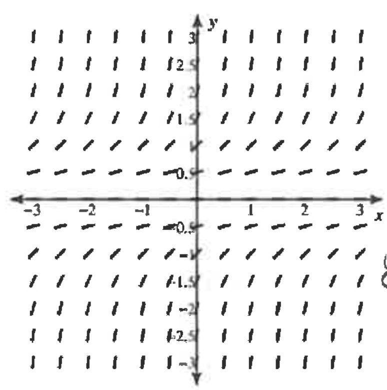
6. Sketch a slope field for the differential equation $y' = 2x + y$ at the indicated points. Sketch the solution that passes through $(1, 1)$.

x	y	dy/dx
0	0	0
0	1	1
1	0	2
1	1	3
2	-1	3
-1	0	-2



7. For each problem, find a differential equation that could be represented with the given slope field.

a.



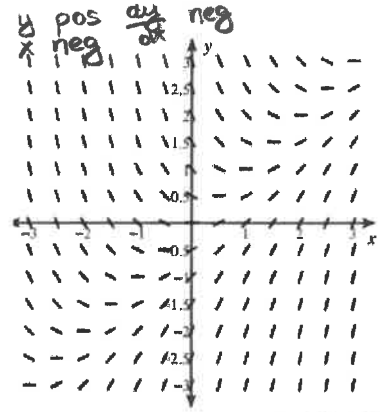
y neg $\frac{dy}{dx} > 0$

~~A)~~ $\frac{dy}{dx} = -\frac{1}{x}$ B) $\frac{dy}{dx} = -\frac{1}{y}$

~~C)~~ $\frac{dy}{dx} = 1$ D) $\frac{dy}{dx} = y^2$

b.

y pos $\frac{dy}{dx}$ neg
x neg



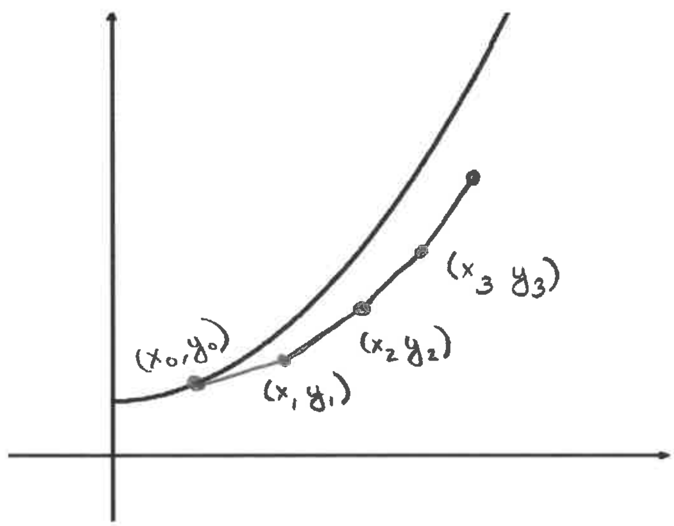
$y = x$
$\frac{dy}{dx} = 0$

x pos
y neg $\frac{dy}{dx}$ pos

A) $\frac{dy}{dx} = x + y$ B) $\frac{dy}{dx} = x - y$

C) $\frac{dy}{dx} = xy$ ~~D)~~ $\frac{dy}{dx} = -xy$

Euler's Method is a numerical approach to approximating the particular solution of a differential equation. Euler's Method uses the concept of local linearity to approximate the shape of the solution curve. Use the slope of the curve to create short, connected line segments.



- Use the slope at x_0 to create a tangent segment.
- The length of the segment is predetermined and called the step-size (h or Δx).
- The next segment (from x_1 to x_2) uses the slope at x_1 to find the next point x_2.
- This process continues until you reach the value that you are trying to find (i.e. approx. f(1.4)).

In general:
The smaller the step size, the better the approximation will be.

The approximation will be an overestimate if the curve is concave down and an underestimate if the curve is concave up.

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$\Delta y = \Delta x \frac{dy}{dx}$

8. Let $y = f(x)$ be the particular solution to the differential equation $y' = x - y$ with initial condition $f(0) = 1$. Use Euler's Method to approximate $f(1.5)$ using 5 equal step sizes.

$\frac{1.5 - 0}{5} = 0.3$

(x, y)	dy/dx	Δx	Δy	$(x+\Delta x, y+\Delta y)$
(0, 1)	−1	0.3	−0.3	(0.3, 0.7)
(0.3, 0.7)	−0.4	0.3	−0.12	(0.6, 0.58)
(0.6, 0.58)	0.2	0.3	0.06	(0.9, 0.586)
(0.9, 0.586)	0.314	0.3	0.0942	(1.2, 0.6802)
(1.2, 0.6802)	0.5198	0.3	0.15594	(1.5, 0.83614)

$f(1.5) \approx 0.83614$

* calc: euler(dy/dx, x, y, { } , y, Δx)

interval

initial condition

9. Given $\frac{dy}{dx} = 2x + y$ and $y(1) = 3$, approximate y(0) using Euler's Method with 4 equal step sizes.

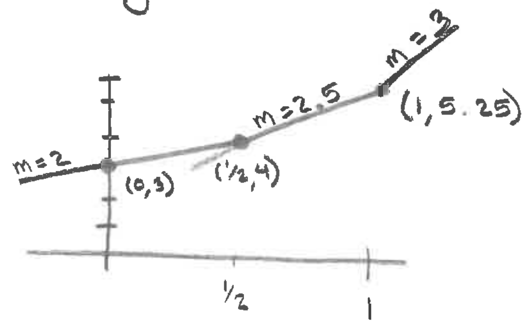
(x, y)	dy/dx	Δx	Δy	$(x+\Delta x,\ y+\Delta y)$
$(1, 3)$	5	-0.25	-1.25	$(0.75, 1.75)$
$(0.75, 1.75)$	3.25	-0.25	-0.8125	$(0.5, 0.9375)$
$(0.5, 0.9375)$	1.9375	-0.25	-0.484375	$(0.25, 0.453125)$
$(0.25, 0.453125)$	0.953125	-0.25	-0.23828125	$(0, 0.21484375)$

$y(0) \approx 0.21484375$

10. Given the differential equation $\frac{dy}{dx} = x + 2$ and $y(0) = 3$. Find an approximation for $y(1)$ by using Euler's method with two equal steps. Sketch your solution.

(x, y)	dy/dx	Δx	Δy	$(x+\Delta x,\ y+\Delta y)$
$(0, 3)$	2	0.5	1	$(0.5, 4)$
$(0.5, 4)$	2.5	0.5	1.25	$(1, 5.25)$

$y(1) \approx 5.25$



a. Solve the differential equation $\frac{dy}{dx} = x + 2$ with the initial condition $y(0) = 3$, and use your solution to find $y(1)$.

$y = \frac{1}{2}x^2 + 2x + C$

$3 = \frac{1}{2}(0)^2 + 2(0) + C$

$3 = C$

$y = \frac{1}{2}x^2 + 2x + 3$

$y(1) = \frac{1}{2}(1)^2 + 2(1) + 3$

$= \frac{1}{2} + 5$

$= 5.5$

b. The error in using Euler's Method is the difference between the approximate value and the exact value. What was the error in your answer? How could you produce a smaller error using Euler's Method?

$E = 5.5 - 5.25$

$= 0.25$

Could use smaller steps (Δx)

11. Given the differential equation $\frac{dy}{dx} = \frac{1}{x+2}$ and $y(0) = 1$, find an approximation of $y(1)$ using Euler's Method with two steps and step size $\Delta x = 0.5$.

(x, y)	dy/dx	Δx	Δy	$(x+\Delta x, y+\Delta y)$
$(0, 1)$	$1/2$	$1/2$	$1/4$	$(1/2, 1.25)$
$(0.5, 1.25)$	$2/5$	$1/2$	$1/5$	$(1, 1.45)$

$y(1) \approx 1.45$

12. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = x + 2y$ with the initial condition $f(0) = 1$. Use Euler's Method, starting at $x = 0$ with two steps of equal size to approximate $f(-0.6)$.

(x, y)	dy/dx	Δx	Δy	$(x+\Delta x, y+\Delta y)$
$(0, 1)$	2	-0.3	-0.6	$(-0.3, 0.4)$
$(-0.3, 0.4)$	0.5	-0.3	-0.15	$(-0.6, 0.25)$

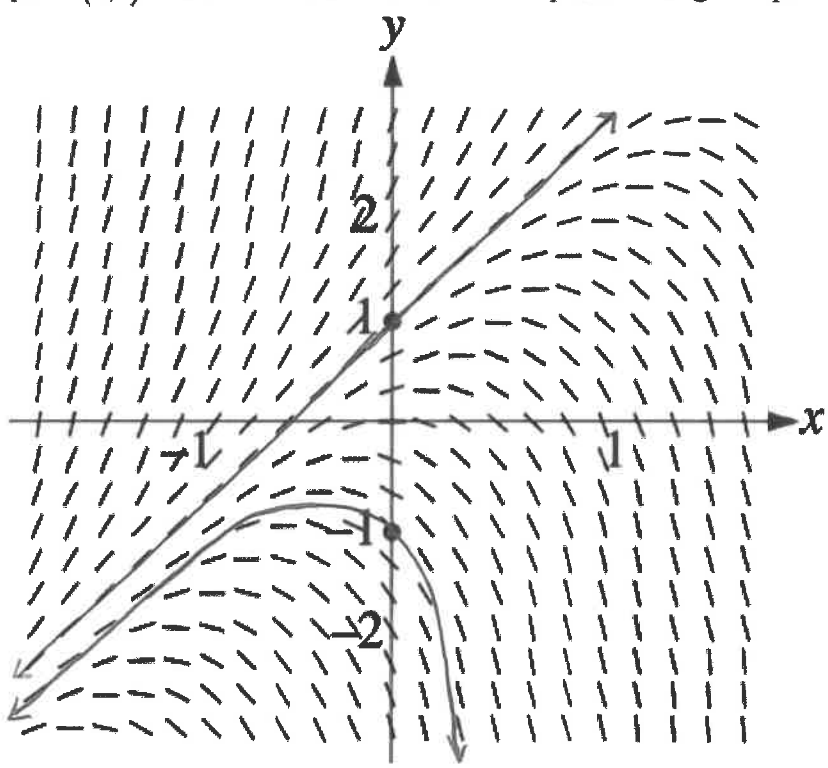
$f(-0.6) \approx 0.25$

13.

AP 2002-5 (No Calculator)

Consider the differential equation: $\frac{dy}{dx} = 2y - 4x$.

(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,1)$ and sketch the solution curve that passes through the point $(0,-1)$.



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(x, y)	dy/dx	Δx	Δy	$(x+\Delta x, y+\Delta y)$
$(0, 1)$	2	0.1	0.2	$(0.1, 1.2)$
$(0.1, 1.2)$	2	0.1	0.2	$(0.2, 1.4)$

$\boxed{f(0.2) \approx 1.4}$

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

$y = 2x + b$

$dy/dx = 2$

$2 = 2y - 4x$

$2 = 2(2x + b) - 4x$

$2 = 4x + 2b - 4x$

$\boxed{1 = b}$

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

$g'(0) = \frac{dy}{dx}\Big|_{(0,0)} = 2(0) - 4(0) = 0$

so $(0,0)$ is a critical pt

$g''(x) = \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4$

$= 2(2y - 4x) - 4$

$= 4y - 8x - 4$

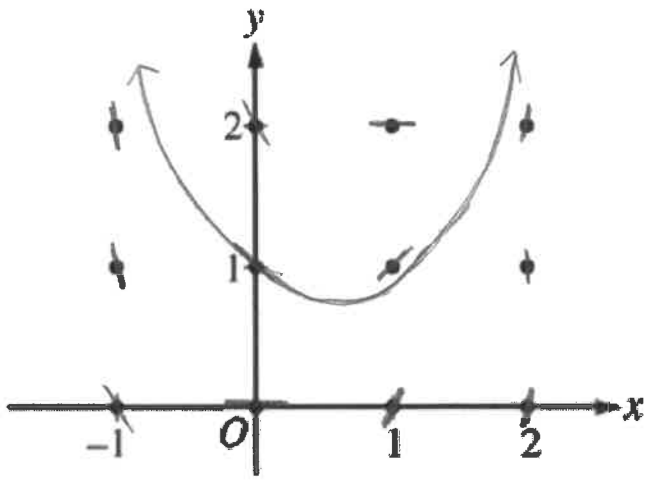
$\frac{d^2y}{dx^2}\Big|_{(0,0)} = -4$ concave down

$\boxed{g \text{ has a local max at } (0,0)}$

14. AP 2005 - 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated and sketch the solution curve that passes through the point $(0,1)$.



(b) The solution curve that passes through the point $(0,1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

$\frac{dy}{dx} = 0$ at $x = \ln 3/2$

$0 = 2(\ln 3/2) - y$

$\boxed{y = \ln 9/4}$

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(x, y)	dy/dx	Δx	Δy	$(x+\Delta x,\ y+\Delta y)$
$(0,1)$	-1	-0.2	0.2	$(-0.2,\ 1.2)$
$(-0.2,\ 1.2)$	-1.6	-0.2	0.32	$(-0.4,\ 1.52)$

$\boxed{f(-0.4) \approx 1.52}$

(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$$

$$= 2 - (2x - y)$$

$$= 2 - 2x + y$$

$f(-0.4) \approx 1.52$ in quadrant II

$y > 0$ and $x < 0$

$\frac{d^2y}{dx^2} > 0$ concave up

the approximation is an underestimate