DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

Properties of Indefinite Integrals

$\int k f(x)dx = k \int f(x)\, dx$ for any constant k

$$\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Power Formulas

$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$ when $n \neq -1$

$\int u^{-1}\, du = \int \frac{1}{u}\, du = \ln|u| + C$ (see Example 2)

Trigonometric Formulas

$$\int \cos u\, du = \sin u + C$$

$$\int \sin u\, du = -\cos u + C$$

$$\int \sec^2 u\, du = \tan u + C$$

$$\int \csc^2 u\, du = -\cot u + C$$

$$\int \sec u \tan u\, du = \sec u + C$$

$$\int \csc u \cot u\, du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\, du = e^u + C$$

$$\int a^u\, du = \frac{a^u}{\ln a} + C$$

$\int \ln u\, du = u \ln u - u + C$ (See Example 2)

$$\int \log_a u\, du = \int \frac{\ln u}{\ln a}\, du = \frac{u \ln u - u}{\ln a} + C$$

★ u-sub undoes chain rule

1. $\int 3x^2(x^3+5)^{20}dx$

$u = x^3+5$

$du = 3x^2\,dx$

$\int u^{20}\,du$

$= \frac{u^{21}}{21} + C$

$= \frac{1}{21}(x^3+5)^{21} + C$

2. $\int (\cos x)^4 \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$-\int u^4\,du$

$= -\frac{1}{5}u^5 + C$

$= -\frac{1}{5}\cos^5 x + C$

3. $\int 10x(5x^2-3)^6 dx$

$u = 5x^2-3$

$du = 10x\,dx$

$\int u^6\,du$

$= \frac{u^7}{7} + C$

$= \frac{1}{7}(5x^2-3)^7 + C$

4. $\int x^4(x^5-12)^7 dx$

$u = x^5-12$

$du = 5x^4\,dx$

$\frac{1}{5}\,du = x^4\,dx$

$\frac{1}{5}\int u^7\,du$

$= \frac{1}{5}\cdot\frac{1}{8}u^8 + C$

$= \frac{1}{40}(x^5-12)^8 + C$

5. $\int x \sin x^2 \, dx$

$u = x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

$\frac{1}{2} \int \sin u \, du$

$= -\frac{1}{2} \cos u + C$

$= -\frac{1}{2} \cos x^2 + C$

7. $\int \frac{\sec^2 x}{\tan x} \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$\int \frac{1}{u} \, du$

$= \ln|u| + C$

$= \ln|\tan x| + C$

6. $\int 3 \sin(3x - 1) dx$

$u = 3x - 1$

$du = 3 \, dx$

$\int \sin u \, du$

$= -\cos u + C$

$= -\cos(3x - 1) + C$

8. $\int x \cos(3x^2 + 1) \, dx$

$u = 3x^2 + 1$

$du = 6x \, dx$

$\frac{1}{6} du = x \, dx$

$\frac{1}{6} \int \cos u \, du$

$= \frac{1}{6} \sin(3x^2 + 1) + C$

9. $\int \frac{4x^6}{(x^7+8)^5}\,dx$

$u = x^7 + 8$

$du = 7x^6\,dx$

$\frac{1}{7}\,du = x^6\,dx$

$\frac{4}{1}\cdot\frac{1}{7}\int \frac{1}{u^5}\,du$

$\frac{4}{7}\left(\frac{1}{-4}u^{-4}\right) + C$

$-\frac{1}{7(x^7+8)^4} + C$

Challenge

10. $\int x^3\sqrt{x^2-6}\,dx$

$= \int x\,x^2\sqrt{x^2-6}\,dx$

$u = x^2 - 6$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$= \frac{1}{2}\int x^2\sqrt{u}\,du$

$u = x^2 - 6$

$u + 6 = x^2$

$= \frac{1}{2}\int (u+6)\sqrt{u}\,du$

$= \frac{1}{2}\int\left(u^{3/2} + 6u^{1/2}\right)du$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} + 6\cdot\frac{2}{3}u^{3/2}\right) + C$

$= \frac{1}{5}u^{5/2} + 2u^{3/2} + C$

$= \frac{1}{5}(x^2-6)^{5/2} + 2(x^2-6)^{3/2} + C$

Challenge!

11. $\int x\sqrt{x+9}\,dx$

$u = x + 9$

$du = dx$

$= \int x\sqrt{u}\,du$

$u = x + 9$

$u - 9 = x$

$= \int (u-9)\sqrt{u}\,du$

$= \int\left(u^{3/2} - 9u^{1/2}\right)du$

$= \frac{2}{5}u^{5/2} - 9\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}u^{5/2} - 6u^{3/2} + C = \frac{2}{5}(x+9)^{5/2} - 6(x+9)^{3/2} + C$

12. $\int_1^2 2x(x^2-2)^3\,dx$

$u = x^2-2$

$du = 2x\,dx$

$= \int_{-1}^{2} u^3\,du$

$= \frac{1}{4}u^4 \Big|_{-1}^{2}$

$= \frac{1}{4}\left[(2)^4 - (-1)^4\right]$

$= \frac{1}{4}(15)$

$= \boxed{15/4}$

13. $\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$

$u = 1-4t^3$

$du = -12t^2\,dt$

$-\frac{1}{6}du = 2t^2\,dt$

$= -\frac{1}{6}\int_{33}^{1}\sqrt{u}\,du$

$= \frac{1}{6}\int_{1}^{33}\sqrt{u}\,du$

$= \frac{1}{6}\left(\frac{2}{3}u^{3/2}\Big|_1^{33}\right)$

$= \frac{1}{9}(33)^{3/2} - \frac{1}{9}(1)^{3/2}$

$= \boxed{\dfrac{(33)^{3/2}-1}{9}} \approx 20.952$

14. $\int_0^{\frac{1}{2}} (e^y + 2\cos(\pi y))\,dy$

$u = \pi y$

$du = \pi\,dy$

$\frac{1}{\pi}du = dy$

$= \int_0^{1/2} e^y\,dy + \frac{2}{\pi}\int_0^{\pi/2}\cos u\,du$

$= e^y\Big|_0^{1/2} + \frac{2}{\pi}\sin u\Big|_0^{\pi/2}$

$= e^{1/2} - e^0 + \frac{2}{\pi}\sin\pi/2 - \frac{2}{\pi}\sin 0$

$= e^{1/2} - 1 + \frac{2}{\pi}$

$= \boxed{\sqrt{e} - 1 + 2/\pi} \approx 1.285$

15. $\int_{-5}^{5} \frac{4t}{2-8t^2}\,dt$

$u = 2-8t^2$

$du = -16t\,dt$

$-\frac{1}{4}du = 4t\,dt$

$= -\frac{1}{4}\int_{198}^{198}\frac{1}{u}\,du$

$\boxed{= 0}$

16. $\int_{0}^{\ln(1+\pi)} e^{x}\cos(1-e^{x})\,dx$

$u = 1 - e^{x}$

$du = -e^{x}\,dx$

$-du = e^{x}\,dx$

Bounds

$x = 0 \quad u = 1 - e^{0} = 1 - 1 = 0$

$x = \ln(1+\pi) \quad u = 1 - e^{\ln(1+\pi)} = 1 - (1+\pi) = -\pi$

$= -\int_{0}^{-\pi} \cos u\,du$

$= \int_{-\pi}^{0} \cos u\,du$

$= \sin u \Big|_{-\pi}^{0}$

$= \sin 0 - \sin(-\pi)$

$= 0 - 0$

$\boxed{= 0}$

Challenge!

17. $\int_{1}^{5} \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x - 1$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

$= \frac{1}{2}\int_{1}^{9} \frac{x}{\sqrt{u}}\,du$

$u = 2x - 1$

$x = \frac{u+1}{2}$

$= \frac{1}{2}\int_{1}^{9} \frac{\frac{u+1}{2}}{\sqrt{u}}\,du$

$= \frac{1}{2}\int_{1}^{9} \frac{u+1}{2} \cdot \frac{1}{\sqrt{u}}\,du$

$= \frac{1}{4}\int_{1}^{9}\left(\frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}\right)du$

$= \frac{1}{4}\int_{1}^{9}\left(u^{1/2} + u^{-1/2}\right)du$

$= \frac{1}{4}\left(\frac{2}{3}u^{3/2} + 2u^{1/2}\right)\Big|_{1}^{9}$

$= \left(\frac{1}{6}\sqrt{9}^{\,3} + \frac{1}{2}\sqrt{9}\right) - \left(\frac{1}{6}\sqrt{1}^{\,3} + \frac{1}{2}\sqrt{1}\right)$

$= \frac{27}{6} + \frac{3}{2} - \frac{1}{6} - \frac{1}{2} = \boxed{16/3}$