

**DEFINITION Indefinite Integral**

The family of all antiderivatives of a function  $f(x)$  is the **indefinite integral of  $f$**  with respect to  $x$  and is denoted by  $\int f(x) dx$ .

If  $F$  is any function such that  $F'(x) = f(x)$ , then  $\int f(x) dx = F(x) + C$ , where  $C$  is an arbitrary constant, called the **constant of integration**.

**Properties of Indefinite Integrals**

$$\int k f(x) dx = k \int f(x) dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

**Power Formulas**

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad \text{when } n \neq -1$$

$$\int u^{-1} du = \int \frac{1}{u} du = \ln |u| + C$$

(see Example 2)

**Trigonometric Formulas**

$$\int \cos u du = \sin u + C$$

$$\int \sin u du = -\cos u + C$$

$$\int \sec^2 u du = \tan u + C$$

$$\int \csc^2 u du = -\cot u + C$$

$$\int \sec u \tan u du = \sec u + C$$

$$\int \csc u \cot u du = -\csc u + C$$

**Exponential and Logarithmic Formulas**

$$\int e^u du = e^u + C$$

$$\int a^u du = \frac{a^u}{\ln a} + C$$

$$\int \ln u du = u \ln u - u + C \quad (\text{See Example 2})$$

$$\int \log_a u du = \int \frac{\ln u}{\ln a} du = \frac{u \ln u - u}{\ln a} + C$$

U-SUB

\*unwinding chain rule

$$\frac{d}{dx} [f(g)] = f'(g) g' \quad \begin{matrix} * \text{looking for inside} \\ \text{function} \end{matrix}$$

1. Evaluate indefinite integral:

$$\begin{aligned} \text{a. } & \int x^{-2} dx \\ & = -x^{-1} + C \end{aligned}$$

$$\begin{aligned} \text{b. } & \int \frac{dt}{t^2+1} \\ & = \tan^{-1}(t) + C \end{aligned}$$

$\int x^{1/2}$

$$\begin{aligned} \text{c. } & \int (2e^x + \sec x \tan x - \sqrt{x}) dx \\ & = 2e^x + \sec x - \frac{2}{3} x^{3/2} + C \end{aligned}$$

$$\begin{aligned} \text{d. } & \int x \cos(2x^2) dx = \int \cos(\underline{2x^2}) \underline{x dx} \\ & u = 2x^2 \end{aligned}$$

$$\frac{du}{dx} = 4x$$

$$du = 4x dx$$

$$\frac{1}{4} du = x dx$$

$$\int \cos(u) \frac{1}{4} du$$

$$\frac{1}{4} \int \cos(u) du$$

$$\frac{1}{4} \sin u + C$$

$$\boxed{\frac{1}{4} \sin(2x^2) + C}$$

$$\begin{aligned} \text{e. } & \int 28(7x-2)^3 dx \\ & u = 7x-2 \end{aligned}$$

$$\frac{du}{dx} = 7$$

$$du = 7dx$$

$$4du = 28dx$$

$$\int u^3 4 du$$

$$4 \int u^3 du$$

$$4 \cdot \frac{u^4}{4} + C$$

$$u^4 + C$$

$$\boxed{f. \int \frac{9r^2 dr}{\sqrt{1-r^3}} \quad (7x-2)^4 + C}$$

$$u = 1-r^3$$

$$\frac{du}{dr} = -3r^2$$

$$-3du = 9r^2 dr$$

$$\int \frac{-3du}{\sqrt{u}}$$

$$-3 \int u^{-1/2} du$$

$$-3(2u^{1/2}) + C$$

$$\boxed{-6(1-r^3)^{1/2} + C}$$

$$g. \int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)dy$$

$$u = y^4 + 4y^2 + 1$$

$$\frac{du}{dy} = 4y^3 + 8y$$

$$du = 4(y^3 + 2y)dy$$

$$2du = 8(y^3 + 2y)dy$$

$$\int u^2 \cdot 2du$$

$$2 \frac{u^3}{3} + C$$

$$\boxed{-\frac{2}{3}(y^4 + 4y^2 + 1)^3 + C}$$

$$h. \int \sec(\theta + \frac{\pi}{2}) \tan(\theta + \frac{\pi}{2}) d\theta$$

$$u = \theta + \frac{\pi}{2}$$

$$\frac{du}{d\theta} = 1$$

$$du = d\theta$$

$$\int \sec u \tan u du$$

$$= \sec u + C$$

$$\boxed{= \sec(\theta + \frac{\pi}{2}) + C}$$

$$i. \int 3(\sin x)^{-2} dx$$

$$3 \int \frac{1}{(\sin x)^2} dx$$

$$3 \int (\csc x)^2 dx$$

$$\boxed{-3 \cot x + C}$$

$$j. \int \sqrt{\cot x} \csc^2 x dx$$

$$u = \cot x$$

$$\frac{du}{dx} = -\csc^2 x$$

$$-du = \csc^2 x dx$$

$$\int \sqrt{u} \cdot (-du)$$

$$-\frac{2}{3} u^{\frac{3}{2}} + C$$

$$\boxed{-\frac{2}{3} (\cot x)^{\frac{3}{2}} + C}$$

$$k. \int \tan^7(\frac{x}{2}) \sec^2(\frac{x}{2}) dx$$

$$u = \tan(\frac{x}{2})$$

$$\frac{du}{dx} = \frac{1}{2} \sec^2(\frac{x}{2})$$

$$du = \frac{1}{2} \sec^2(\frac{x}{2}) dx$$

$$2 du = \sec^2(\frac{x}{2}) dx$$

$$\int u^7 \cdot 2 du$$

$$2 \int u^7 du$$

$$2 \cdot \frac{u^8}{8} + C$$

$$\boxed{\frac{1}{4} (\tan(\frac{x}{2}))^8 + C}$$

$$\text{l. } \int \frac{dx}{\sin^2 3x}$$

$$\int \csc^2(3x) dx$$

$$u = 3x$$

$$\frac{du}{dx} = 3$$

$$du = 3dx$$

$$\frac{1}{3} du = dx$$

$$\frac{1}{3} \int \csc^2 u du$$

$$- \frac{1}{3} \cot u + C$$

$$\boxed{- \frac{1}{3} \cot(3x) + C}$$

$$\text{m. } \int \frac{6 \cos t}{(2+\sin t)^2} dt$$

$$u = 2 + \sin t$$

$$\frac{du}{dt} = \cos t$$

$$du = \cos t dt$$

$$6du = 6 \cos t dt$$

$$\int \frac{1}{u^2} 6du$$

$$6 \int u^{-2} du$$

$$-6u^{-1} + C$$

$$\boxed{-6(2+\sin t)^{-1} + C}$$

$$\text{n. } \int \tan^2 x \sec^2 x dx$$

$$u = \tan x$$

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x dx$$

$$\int u^2 du$$

$$\frac{u^3}{3} + C$$

$$\boxed{\frac{(\tan x)^3}{3} + C}$$

$$\text{o. } \int \frac{40}{x^2+25} dx$$

$$\frac{40}{25} \int \frac{1}{(x/5)^2 + 1} dx$$

$$\frac{40}{25} \int \frac{1}{(x/5)^2 + 1} dx$$

$$u = x/5$$

$$\frac{du}{dx} = \frac{1}{5}$$

$$du = \frac{1}{5} dx$$

$$5du = dx$$

$$\frac{40}{25} \int \frac{1}{u^2 + 1} 5du$$

$$\frac{40}{5} \tan^{-1} u + C$$

$$\boxed{8 \tan^{-1}(x/5) + C}$$

## AB Calculus

## 7.2 Antidifferentiation by Substitution

2. Make a u-substitution and integrate from  $u(a)$  to  $u(b)$

a.  $\int_0^1 r\sqrt{1-r^2} dr$

$$u = 1 - r^2$$

$$\frac{du}{dr} = -2r$$

$$-\frac{1}{2} du = r dr$$

$$\begin{aligned} & -\frac{1}{2} \int_{1-0^2}^{1-1^2} u^{\frac{1}{2}} du = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_0^1 \\ & = -\frac{1}{2} \int_1^0 u^{\frac{1}{2}} du = \frac{1}{3} (1)^{\frac{3}{2}} - \frac{1}{3} (0)^{\frac{3}{2}} \\ & = \frac{1}{2} \int_0^1 u^{\frac{1}{2}} du = \boxed{\frac{1}{3}} \end{aligned}$$

b.  $\int_{-1}^1 \frac{5r}{(4+r^2)^2} dr$

$$u = 4 + r^2$$

$$du = 2r dr$$

$$\frac{1}{2} du = r dr$$

$$\frac{5}{2} \int_5^5 u^{-2} du$$

$$= \boxed{0}$$

c.  $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3 \sin x}} dx$

$$\begin{aligned} 4 + 3 \sin \pi &= 4 \\ 4 + 3 \sin(-\pi) &= 4 \end{aligned}$$

$$u = 4 + 3 \sin x$$

$$du = 3 \cos x dx$$

$$\frac{1}{3} du = \cos x dx$$

$$\begin{aligned} & \frac{1}{3} \int_4^4 u^{-\frac{1}{2}} du \\ & \boxed{= 0} \end{aligned}$$



d.  $\int_0^{\frac{\pi}{6}} \cos^{-3} 2\theta \sin 2\theta d\theta$

$$\begin{aligned} \cos \frac{\pi}{3} &= \frac{1}{2} \\ \cos 0 &= 1 \end{aligned}$$

$$u = \cos 2\theta$$

$$du = -2 \sin 2\theta d\theta$$

$$-\frac{1}{2} du = \sin 2\theta d\theta$$

$$-\frac{1}{2} \int_1^{\frac{1}{2}} u^{-3} du$$

$$\frac{1}{2} \int_{\frac{1}{2}}^1 u^{-3} du$$

$$\frac{1}{2} \left. \frac{u^{-2}}{-2} \right|_{\frac{1}{2}}^1$$

$$-\frac{1}{4} (1)^{-2} + \frac{1}{4} (\frac{1}{2})^{-2}$$

$$-\frac{1}{4} + \frac{1}{4} (4)$$

$$\boxed{\frac{3}{4}}$$

AB Calculus  
7.2 Antidifferentiation by Substitution

e.  $\int_2^5 \frac{dx}{2x-3}$

$$u = 2x - 3$$

$$du = 2 dx$$

$$\frac{1}{2} du = dx$$

$$\frac{1}{2} \int_1^7 u^{-1} du$$

$$\frac{1}{2} \ln|u| \Big|_1^7$$

$$\frac{1}{2} \ln|7| - \frac{1}{2} \ln|1|$$

$\frac{\ln 7}{2}$

f.  $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x dx$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\cos x}{\sin x} dx$$

$$u = \sin x$$

$$du = \cos x dx$$

$$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}$$

$$\int u^{-1} du$$

$$\int_{\frac{\sqrt{2}}{2}}$$

$= 0$

g.  $\int_0^2 \frac{e^x dx}{3+e^x}$

$$u = 3 + e^x$$

$$du = e^x dx$$

$$3+e^2$$

$$\int u^{-1} du$$

$$4$$

$$\ln|u| \Big|_4^{3+e^2}$$

$$\ln|3+e^2| - \ln|4|$$

$\ln \frac{3+e^2}{4}$

AB Calculus  
7.2 Antidifferentiation by Substitution

3. Use the given trigonometric identity to set up a u-substitution and then evaluate the indefinite integral.

a.  $\int \sec^4 x dx$ ,  $\sec^2 x = 1 + \tan^2 x$

$$\int (1 + \tan^2 x) \sec^2 x dx$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

$$\int (1 + u^2) du$$

$$u + \frac{1}{3}u^3 + C$$

$$\boxed{\tan x + \frac{1}{3}\tan^3 x + C}$$

b.  $\int 4\cos^2 x dx$ ,  $\cos 2x = 2\cos^2 x - 1$

$$\cos 2x + 1 = 2\cos^2 x$$

$$2 \int (\cos 2x + 1) dx$$

$$u = 2x$$

$$du = 2 dx \quad \frac{1}{2} du = dx$$

$$2 \int (\cos u + 1) \frac{1}{2} du$$

$$\int (\cos u + 1) du$$

$$\sin u + u + C$$

$$\boxed{\sin 2x + 2x + C}$$

