$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

*inverse product rule

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$-\int v\frac{du}{dx}dx \qquad -\int v\frac{du}{dx}dx$$

$$uv - \int v\frac{du}{dx}dx = \int u\frac{dv}{dx}dx$$

$$uv - \int v\,du = \int u\,dv$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

set u to:

L og

I nverse trig

P olynomial

E xponential

T rig

1. Find the indefinite integral:

assign u and v
* taking derivative of u so want derivative of u to be simpler
* taking antiderivative of v so want $\int v$ to be simpler

a. $\int xe^x\,dx$

$u = x \qquad v = e^x$

$du = 1dx \qquad dv = e^x dx$

$$\int xe^x dx = xe^x - \int e^x(1)\,dx$$

$$\boxed{= xe^x - e^x + C}$$

b. $\int 2t\cos(3t)\,dt$

$u = 2t \qquad v = \frac{1}{3}\sin 3t$

$du = 2\,dt \qquad dv = \cos 3t\,dt$

$\int \cos 3t\,dt$

$u = 3t$

$du = 3\,dt$

$\frac{1}{3}\int \cos u\,dt$

$\frac{1}{3}\sin 3t$

$$\int 2t\cos(3t)\,dt = 2t\left(\tfrac{1}{3}\sin 3t\right) - \int \tfrac{1}{3}\sin 3t\,(2\,dt)$$

$$= \frac{2}{3}t\sin 3t - \frac{2}{3}\int \sin 3t\,dt$$

$u = 3t$

$du = 3dt$

$$= \frac{2}{3}t\sin 3t - \frac{2}{9}\int \sin u\,du$$

$$\boxed{= \frac{2}{3}t\sin 3t + \frac{2}{9}\cos 3t + C}$$

c. $\int x^2e^{-x}\,dx$

$u = x^2$ $\quad v = -e^{-x}$

$du = 2x\,dx$ $\quad dv = e^{-x}\,dx$

$$\int x^2 e^{-x}\,dx = -x^2e^{-x} + \int e^{-x}\,2x\,dx$$

$u = 2x$ $\quad V = -e^{-x}$

$du = 2\,dx$ $\quad dv = e^{-x}dx$

$$= -x^2e^{-x}\left[-2xe^{-x} + \int 2e^{-x}\,dx\right]$$

$$\boxed{= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C}$$

d. $\int t^2 \ln t\,dt$

$u = \ln t$ $\quad v = \frac{t^3}{3}$

$du = \frac{1}{t}\,dt$ $\quad dv = t^2\,dt$

$$\int t^2 \ln t\,dt = \frac{\ln t\,(t^3)}{3} - \int \frac{t^3}{3}\frac{1}{t}\,dt$$

$$= \frac{1}{3}t^3\ln t - \frac{1}{3}\int t^2\,dt$$

$$= \frac{1}{3}t^3\ln t - \frac{1}{3}\frac{t^3}{3} + C$$

$$\boxed{= \frac{1}{3}t^3\ln t - \frac{1}{9}t^3 + C}$$

e. $\int \tan^{-1}x\,dx$

$u = \tan^{-1}x$ $\quad v = x$

$du = \frac{1}{1+x^2}\,dx$ $\quad dv = dx$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\int u^{-1}\,du$$

$$\boxed{= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C}$$

2. Solve the initial value problem:

a. $\frac{dy}{dx} = 2xe^{-x}$ and $y = 3$ when $x = 0$

$y = \int 2xe^{-x}\,dx$

$u = 2x \quad v = -e^{-x}$

$du = 2dx \quad dv = e^{-x}\,dx$

$y = -2xe^{-x} + 2\int e^{-x}\,dx$

$= -2xe^{-x} - 2e^{-x} + C$

$3 = -2(0)e^{0} - 2e^{0} + C$

$3 = -2 + C$

$5 = C$

$\boxed{y = -2xe^{-x} - 2e^{-x} + 5}$

b. $\frac{dy}{dx} = 2x\sqrt{x+2}$ and $y = 0$ when $x = -1$

$y = \int 2x\sqrt{x+2}\,dx$

$u = 2x \quad v = \frac{2}{3}(x+2)^{3/2}$

$du = 2dx \quad dv = \sqrt{x+2}\,dx$

$u = x+2$

$du = 1\,dx$

$\int u^{1/2}\,du$

$\frac{2}{3}u^{3/2}$

$y = \frac{4}{3}x(x+2)^{3/2} - \frac{4}{3}\int (x+2)^{3/2}\,dx$

$u = x+2 \quad du = dx$

$\int u^{3/2}\,du$

$= \frac{4}{3}x(x+2)^{3/2} - \frac{8}{15}(x+2)^{5/2} + C$

$0 = \frac{4}{3}(3)^{3/2} - \frac{8}{15}(3)^{5/2} + C$

$\frac{28}{15} = C$

$\boxed{y = \frac{4}{3}x(x+2)^{3/2} - \frac{8}{15}(x+2)^{5/2} + \frac{28}{15}}$

c. $\frac{dy}{dx} = x^3 \ln x$ and $y = 5$ when $x = 1$

$y = \int x^3 \ln x\,dx$

$u = \ln x \quad v = \frac{1}{4}x^4$

$du = \frac{1}{x}dx \quad dv = x^3\,dx$

$y = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^4/x\,dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\cdot\frac{1}{4}x^4 + C$

$5 = \frac{1}{4}\ln(1) - \frac{1}{16} + C$

$\frac{81}{16} = C$

$\boxed{y = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + \frac{81}{16}}$

3. Use parts and solve the unknown integral:

a. $\int (x^2 - 5x)e^x \, dx$

$u = x^2 - 5x \qquad v = e^x$

$du = 2x - 5dx \qquad dv = e^x \, dx$

$= (x^2 - 5x)e^x - \int (2x - 5)e^x \, dx$

$u = 2x - 5 \qquad v = e^x$

$du = 2\,dx \qquad dv = e^x \, dx$

$= (x^2 - 5x)e^x - \left[(2x-5)e^x - 2\int e^x \, dx\right]$

$= (x^2 - 5x)e^x - (2x-5)e^x + 2e^x + C$

$= e^x(x^2 - 5x - 2x + 5 + 2) + C$

$\boxed{= e^x(x^2 - 7x + 7) + C}$

b. $\int e^{-x} \cos 2x \, dx$

$u = e^{-x} \qquad v = \frac{1}{2}\sin 2x$

$du = -e^{-x}\,dx \qquad dv = \cos 2x \, dx$

$= \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\int e^{-x}\sin 2x \, dx$

$u = e^{-x} \qquad v = -\frac{1}{2}\cos 2x$

$du = -e^{-x} \qquad dv = \sin 2x \, dx$

$= \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\left[-\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x \, dx\right]$

$\underline{\int e^{-x}\cos 2x \, dx} = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x \underline{- \frac{1}{4}\int e^{-x}\cos 2x \, dx}$

* add

$\frac{5}{4}\int e^{-x}\cos 2x \, dx = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x$

$\boxed{\int e^{-x}\cos 2x \, dx = \frac{2}{5}e^{-x}\sin 2x - \frac{1}{5}e^{-x}\cos 2x + C}$

c. $\int_{-3}^{2} e^{-2x} \sin 2x \, dx$

$u = \sin 2x \qquad v = -\frac{1}{2} e^{-2x}$

$du = 2\cos 2x \, dx \qquad dv = e^{-2x} \, dx$

$= -\frac{1}{2} e^{-2x} \sin 2x \Big|_{-3}^{2} + \int_{-3}^{2} \frac{1}{2} e^{-2x} \, 2\cos 2x \, dx$

$= -\frac{1}{2} e^{-4} \sin 4 + \frac{1}{2} e^{6} \sin(-6) + \int_{-3}^{2} e^{-2x} \cos 2x \, dx$

$u = \cos 2x \qquad v = -\frac{1}{2} e^{-2x}$

$du = -2\sin 2x \, dx \qquad dv = e^{-2x} \, dx$

$= -\frac{1}{2} e^{-4} \sin 4 + \frac{1}{2} e^{6} \sin(-6) + \left[-\frac{1}{2} e^{-2x} \cos 2x \Big|_{-3}^{2} - \int_{-3}^{2} e^{-2x} \sin 2x dx \right]$

$= -\frac{1}{2} e^{-4} \sin 4 + \frac{1}{2} e^{6} \sin(-6) - \frac{1}{2} e^{-4} \cos 4 + \frac{1}{2} e^{6} \cos(-6) - \int_{-3}^{2} e^{-2x} \sin 2x \, dx$

$\int_{-3}^{2} e^{-2x} \sin 2x \, dx = -\frac{1}{2} e^{-4} \sin 4 + \frac{1}{2} e^{6} \sin(-6) - \frac{1}{2} e^{-4} \cos 4 + \frac{1}{2} e^{6} \cos(-6) - \int_{-3}^{2} e^{-2x} \sin 2x dx$

$+ \int_{-3}^{2} e^{-2x} \sin 2x \, dx \qquad\qquad + \int_{-3}^{2} e^{-2x} \sin 2x \, dx$

$2\int_{-3}^{2} e^{-2x} \sin 2x \, dx = -\frac{1}{2} e^{-4} \sin 4 + \frac{1}{2} e^{6} \sin(-6) - \frac{1}{2} e^{-4} \cos 4 + \frac{1}{2} e^{6} \cos(-6)$

$\int_{-3}^{2} e^{-2x} \sin 2x \, dx = -\frac{1}{4} e^{-4} \sin 4 + \frac{1}{4} e^{6} \sin(-6) - \frac{1}{4} e^{-4} \cos 4 + \frac{1}{4} e^{6} \cos(-6)$

$= 125.028$

4. **Solve the differential equation:**

a. $\frac{dy}{dx} = x^2 \ln x$

$y = \int x^2 \ln x \, dx$

$u = \ln x \qquad v = \frac{x^3}{3}$

$du = \frac{1}{x} dx \qquad dv = x^2 dx$

$y = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^3 \frac{1}{x} dx$

$y = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$

$y = \frac{1}{3}x^3 \ln x - \frac{1}{3}\frac{x^3}{3} + C$

$\boxed{y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$

b. $\frac{dy}{d\theta} = \theta \sec\theta \tan\theta$

$y = \int \theta \sec\theta \tan\theta \, d\theta$

$u = \theta \qquad v = \sec\theta$

$du = d\theta \qquad dv = \sec\theta \tan\theta \, d\theta$

$y = \theta \sec\theta - \int \sec\theta \, d\theta$

$y = \theta \sec\theta - \int \sec\theta \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} d\theta$

$u = \sec\theta + \tan\theta$

$du = \sec\theta \tan\theta + \sec^2\theta \, d\theta$

$du = \sec\theta(\sec\theta + \tan\theta) \, d\theta$

$y = \theta \sec\theta - \int \frac{1}{u} du$

$y = \theta \sec\theta - \ln(\sec\theta + \tan\theta) + C$