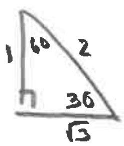
Cosine Sum and Difference Identities

$\cos(A-B)$ ***does not equal*** $\cos A - \cos B$.

For example, if $A = \frac{\pi}{2}$ and $B = 0$, then

$$\cos(A-B) = \cos(\pi/2 - 0) = \cos\frac{\pi}{2} = \underline{0},$$

while $\cos A - \cos B = \cos\frac{\pi}{2} - \cos \underline{0} = \underline{0} - \underline{1} = \underline{-1}$.



Cosine of a Sum or Difference

$\cos(A+B) = \underline{\cos A \cos B - \sin A \sin B}$

$\cos(A-B) = \underline{\cos A \cos B + \sin A \sin B}$

S | A
T | C

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

CLASSROOM EXAMPLE 1 Finding Exact Cosine Function Values
Find the *exact* value of each expression.

(a) $\cos(-75°)$

$A = -45$
$B = 30$

$= \cos(-45 - 30)$

$= \cos(-45)\cos(30) + \sin(-45)\sin(30)$

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

(b) $\cos\frac{17\pi}{12} = \cos(255°)$

$= \cos(300 - 45)$

$= \cos 300 \cos 45 + \sin 300 \sin 45$

$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{-\sqrt{3}}{2}\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{4} + \frac{-\sqrt{6}}{4}$

$= \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

(c) $\cos 173° \cos 83° + \sin 173° \sin 83°$

$= \cos(173 - 83)$

$= \cos(90°)$

$= \boxed{0}$

Cofunction Identities

Cofunction Identities

The following identities hold for any angle θ for which the functions are defined.

$\cos(90° - \theta) =$ sin θ $\cot(90° - \theta) =$ tan θ

$\sin(90° - \theta) =$ cos θ $\sec(90° - \theta) =$ csc θ

$\tan(90° - \theta) =$ cot θ $\csc(90° - \theta) =$ sec θ

The same identities can be obtained for a real number domain by replacing 90° with $\frac{\pi}{2}$.

CLASSROOM EXAMPLE 2 Using Cofunction Identities to Find θ

Find one value of θ or x that satisfies each of the following.

(a) $\sec\theta = \csc 62°$ **(b)** $\tan\theta = \cot(-54°)$ **(c)** $\cos x = \sin\frac{7\pi}{6}$

(a)
$\csc(90 - \theta) =$
$90 - \theta = 62$
$\boxed{\theta = 28°}$

(b)
$\tan(90 - \theta) =$
$90 - \theta = -54$
$\boxed{\theta = 144°}$

(c)
$\sin(2\pi - x) =$
$2\pi - x = \frac{7\pi}{6}$
$x = 2\pi - \frac{7\pi}{6}$
$= \frac{12\pi}{6} - \frac{7\pi}{6}$
$= \boxed{\frac{5\pi}{6}}$

Sine and Tangent Sum and Difference Identities

Sine of a Sum or Difference

$\sin(A + B) =$ $\sin A\cos B + \cos A\sin B$

$\sin(A - B) =$ $\sin A\cos B - \cos A\sin B$

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

Tangent of a Sum or Difference

$\tan(A + B) =$ $\frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) =$ $\frac{\tan A - \tan B}{1 + \tan A\tan B}$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Applications of the Sum and Difference Identities

CLASSROOM EXAMPLE 3 Finding Exact Sine and Tangent Function Values
Find the *exact* value of each expression.

(a) $\sin(-15°)$

$= \sin(30 - 45)$

$= \sin 30 \cos 45 - \sin 45 \cos 30$

$= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

(b) $\tan\frac{13\pi}{12} = \tan 195$

$= \tan(225 - 30)$

$= \frac{\tan 225 - \tan 30}{1 + \tan 225 \tan 30}$

$= \frac{1 - \sqrt{3}/3}{1 + (1)(\sqrt{3}/3)}$

$= \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}$

$= \frac{3-\sqrt{3}}{3+\sqrt{3}}$

$= \frac{(3-\sqrt{3})(3-\sqrt{3})}{9-3}$

$= \frac{9-6\sqrt{3}-3}{6}$

$= \frac{6-6\sqrt{3}}{6}$

$= \boxed{1-\sqrt{3}}$

(c) $\frac{\tan 100° - \tan 70°}{1 + \tan 100° \tan 70°}$

$= \tan(100 - 70)$

$= \tan 30$

$= \boxed{\frac{\sqrt{3}}{3}}$

CLASSROOM EXAMPLE 4 Writing Functions as Expressions Involving Functions of θ
Write each function as an expression involving functions of θ alone.

(a) $\cos(\theta - 270°)$

$= \cos\theta \cos 270 + \sin\theta \sin 270$

$= \cos\theta\ (0) + \sin\theta\ (-1)$

$= \boxed{-\sin\theta}$

(b) $\tan(\theta + 3\pi)$

$= \frac{\tan\theta + \tan 3\pi}{1 - \tan\theta \tan 3\pi}$

$= \frac{\tan\theta + 0}{1 - \tan\theta\,(0)}$

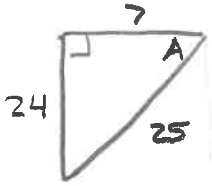
$= \boxed{\tan\theta}$

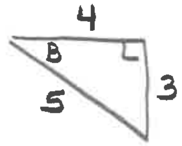
(c) $\sin(120° + \theta)$

$= \sin 120 \cos\theta - \sin\theta \cos 120$

$= \frac{\sqrt{3}}{2}\cos\theta - \sin\theta\ \frac{1}{2}$

$= \boxed{\frac{\sqrt{3}\cos\theta - \sin\theta}{2}}$





CLASSROOM EXAMPLE 5 Finding Function Values and the Quadrant of $A+B$

Suppose that A and B are angles in standard position such that $\cos A = -\frac{7}{25}$, $\pi < A < \frac{3\pi}{2}$, and $\sin B = -\frac{3}{5}$, $\frac{3\pi}{2} < B < 2\pi$. Find each of the following.

(a) $\sin(A-B)$

$$= \sin A \cos B - \sin B \cos A$$

$$= \frac{-24}{25} \cdot \frac{4}{5} - \frac{-3}{5} \cdot \frac{-7}{25}$$

$$= \frac{-96}{125} - \frac{21}{125} = \boxed{\frac{-117}{125}}$$

(b) $\tan(A-B)$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{24/7 + 3/4}{1 + (24/7)(-3/4)}$$

$$= \frac{117/28}{1 - 72/28}$$

$$= \frac{117/28}{-44/28}$$

$$= \boxed{-\frac{117}{44}}$$

(c) the quadrant of $A-B$

$\sin(A-B)$ neg

$\tan(A-B)$ neg

S	A
T	C

$\boxed{\text{4th quadrant}}$

Verifying an Identity

CLASSROOM EXAMPLE 7 Verifying an Identity
Verify that the equation is an identity.

$$\tan\left(\frac{\pi}{4}+t\right)+\tan\left(\frac{\pi}{4}-t\right)=\frac{2\sec^2 t}{1-\tan^2 t}$$

$$\frac{\tan \pi/4 + \tan t}{1 - \tan \pi/4 \tan t} + \frac{\tan \pi/4 - \tan t}{1 + \tan \pi/4 \tan t} =$$

$$\frac{(\tan \pi/4 + \tan t)(1 + \tan \pi/4 \tan t) + (\tan \pi/4 - \tan t)(1 - \tan \pi/4 \tan t)}{1 - (\tan \pi/4 \tan t)^2} =$$

$$\frac{(1+\tan t)(1+\tan t) + (1-\tan t)(1-\tan t)}{1-\tan^2 t} =$$

$$\frac{1 + 2\tan t + \tan^2 t + 1 - 2\tan t + \tan^2 t}{1-\tan^2 t} =$$

$$\frac{2 + 2\tan^2 t}{1-\tan^2 t} =$$

$$\frac{2\sec^2 t}{1-\tan^2 t} =$$