1. Direct proportion equation : $y = kx$
2. Inverse (indirect) proportion equation: $y = \frac{k}{x}$
3. k is called the constant of proportionality

Exponential Growth/Decay class examples

1) If the rate of change of y varies directly with the value of y, find the general equation:

$y' = ky$

$\frac{dy}{dt} = ky$

* separate & integrate

$\frac{1}{y}\frac{dy}{dt} = k$

$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt$

$\ln|y| = kt + C$

$y = e^{kt+C}$

$y = e^{kt}e^{C}$

$\boxed{y = Ce^{kt}}$

2) The rate of increase of the population of a certain city is proportional to the population. If the population in 1930 was 50,000 and in 1960 it was 75,000, what was the expected population in 1990?

let $t = 0$ represent 1930

$t = 30 \rightarrow 1960$

$t = 60 \rightarrow 1990$

P = population

$P' = kP$

$\frac{dP}{dt} = kP$

$\frac{1}{P}\frac{dP}{dt} = k$

$\int \frac{1}{P}\frac{dP}{dt}\,dt = \int k\,dt$

$\ln|P| = kt + C$

$P = e^{kt+C}$

$P = Ce^{kt}$

$(0, 50{,}000)$

$(30, 75{,}000)$

$(60, \_\_\_\_)$

use given to solve for C, then k

$50{,}000 = Ce^{k0}$

$50{,}000 = C$

* C always initial

$P = 50{,}000e^{kt}$

$75{,}000 = 50{,}000e^{30k}$

$1.5 = e^{30k}$

$\ln 1.5 = 30k$

$\frac{\ln 1.5}{30} = k$

$P = 50{,}000e^{\frac{\ln 1.5}{30}t}$

$P = 50{,}000e^{60\left(\frac{\ln 1.5}{30}\right)}$

$\boxed{P = 112{,}500}$

3) The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

$(\overset{t}{\text{year}}, \text{radium} \overset{r}{\text{amount}})$

$(0, 60)$

$(100, \_\_)$

$(1690, 30)$

$$\frac{dr}{dt} = Kr$$

$$\int \frac{1}{r}\frac{dr}{dt}\,dt = \int K\,dt$$

$$\ln|r| = Kt$$

$$r = Ce^{Kt}$$

using $(0, 60)$

$$60 = Ce^{K(0)}$$

$$60 = C$$

$$r = 60e^{Kt}$$

using $(1690, 30)$

$$30 = 60e^{K(1690)}$$

$$\frac{1}{2} = e^{1690K}$$

$$\frac{\ln 1/2}{1690} = K$$

$$r = 60e^{\frac{\ln 1/2}{1690}t}$$

$$r = 60e^{\frac{\ln 1/2}{1690}(100)}$$

$$\boxed{r = 57.589 \text{ mg}}$$

4. In a certain culture where the rate of growth of bacteria is proportional to the amount present, the number triples in 3 hours.
A) If at the end of 12 hours there were 10 million bacteria, how many were present initially?
B) Find the specific exponential growth equation

A)

b = bacteria

$(\overset{t}{\text{time}}, \overset{b}{\text{bacteria}})$

$(0, c)$

$(3, 3c)$

$(12, 10)$

$$\frac{db}{dt} = Kb$$

$$\int \frac{1}{b}\frac{db}{dt}\,dt = \int K\,dt$$

$$\ln|b| = Kt + C$$

$$b = Ce^{Kt}$$

$$3C = Ce^{K(3)}$$

$$3 = e^{3K}$$

$$\frac{\ln 3}{3} = K$$

$$b = Ce^{\frac{1}{3}\ln 3\, t}$$

$$10 = Ce^{\frac{1}{3}\ln 3\,(12)}$$

$$10 = Ce^{4\ln 3}$$

$$10 = Ce^{\ln 3^4}$$

$$10 = C(3^4)$$

$$\frac{10}{3^4} = C$$

$$0.123 \approx C$$

$\boxed{\text{* initially } 0.123 \text{ mill. bacteria or } 123{,}000 \text{ bacteria}}$

B)

$$b = \frac{10}{3^4}e^{\frac{1}{3}\ln 3\, t}$$

Newton's Law of Cooling: The rate of change in temperature of an object is proportional to the difference between the object's temperature and temperature of the surrounding medium.

surrounding temp

5) When an object is removed from an oven and placed in constant 80° F, the core temperature is 1500° F. One hour later the core temperature is 1120° F, find the core temperature 5 hours later.

(time, temp) — (t, y)

(0, 1500)
(1, 1120)
(5, ___)

$$\frac{dy}{dt} = k(y-80)$$

$$\frac{1}{y-80}\frac{dy}{dt} = k$$

$$\int \frac{1}{y-80}\frac{dy}{dt}\,dt = \int k\,dt$$

$$\ln|y-80| = kt + c$$

$$y - 80 = e^{kt+c}$$

$$y - 80 = Ce^{kt}$$

$$y = Ce^{kt} + 80$$

using (0, 1500)

$$1500 = Ce^{k(0)} + 80$$

$$1420 = C$$

$$y = 1420e^{kt} + 80$$

using (1, 1120)

$$1120 = 1420e^{k(1)} + 80$$

$$\frac{1040}{1420} = e^{k}$$

$$\ln 52/71 = k$$

$$y = 1420e^{\ln 52/71\, t} + 80$$

$$y = 1420e^{\ln 52/71\,(5)} + 80$$

$$\boxed{y = 379.236^{\circ}}$$

Extra Practice:

1. Rate of change of y is proportional to y. When $x = 0$, $y = 4$ and when $x = 3$, $y = 10$. Find the value of y when $x = 6$.

$$\frac{dy}{dx} = ky$$

$$\int \frac{1}{y} \frac{dy}{dx}\, dx = \int k\, dx$$

$$\ln|y| = kx + C$$

$$y = Ce^{kx}$$

$(0,4) \rightarrow C = 4$

$(3,10)$

$(6, \_)$

$$y = 4e^{kx}$$

$$10 = 4e^{3k}$$

$$\frac{\ln 10/4}{3} = k$$

$$y = 4e^{\frac{1}{3}\ln 5/2\, t}$$

$$y = 4e^{(1/3 \ln 5/2)(6)}$$

$$\boxed{y = 25}$$

2. Rate of change of V is proportional to V. When $t = 0$, $V = 20,000$ and when $t = 4$, $V = 12,500$. Find the value of V when $t = 6$.

$$\frac{dV}{dt} = kV$$

$$\int \frac{1}{V} \frac{dV}{dt}\, dt = \int k\, dt$$

$$\ln|V| = kt + C$$

$$V = Ce^{kt}$$

(t, V)

$(0, 20000) \rightarrow C = 20,000$

$(4\ 12500)$

$(6, \_)$

$$V = 20,000e^{kt}$$

$$12\,500 = 20,000e^{4k}$$

$$\frac{\ln(0.625)}{4} = k$$

$$V = 20,000e^{\frac{\ln(0.625)}{4}t}$$

$$V = 20,000e^{\frac{1}{4}\ln(0.625)(6)}$$

$$\boxed{V = 9,882.118}$$

3. Radium has a half-life of 1,599 years. Given that after 10,000 years only 0.5 g remain, find the:
   a. Initial quantity
   b. Amount after 10,000 years

(10,000, 0.5)

(0, C)

(1,599, ½C)

$\frac{dy}{dt} = ky$

$y = Ce^{kt}$

$\frac{1}{2}C = Ce^{1599k}$

$\frac{\ln \frac{1}{2}}{1599} = k$

$y = Ce^{\frac{1}{1599}\ln 0.5\, t}$

$0.5 = Ce^{1/1599 \ln 0.5\,(10{,}000)}$

$0.5 = (0.013103)C$

$\boxed{38.159 = C}$

$\boxed{y = 38.159e^{1/1599 \ln 1/2\, t}}$

4. a) Find the growth model for Bulgaria's population given that population in 2001 is 7.7 million people. Let $t = 0$ represent year 2000.

(1, 7.7)

and $k = -0.09$

b) Predict population in 2015

c) as time ↑ what happens w/ population

$P = Ce^{-0.09t}$

$7.7 = Ce^{-0.09(1)}$

$C = 7.7696$

a) $P = 7.769e^{-0.09t}$

b) $t = 15$

$P = 7.769e^{-0.09(15)}$

$P = 6.788$ million

c) since $k < 0$, the pop is decreasing