DEFINITION Separable Differential Equation

A differential equation of the form $dy/dx = f(y)g(x)$ is called **separable.** We **separate the variables** by writing it in the form

$$\frac{1}{f(y)}\,dy = g(x)\,dx.$$

The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

1. Use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

a. $\frac{dy}{dx} = -\frac{x}{y}$ and $y = 3$ when $x = 4$

$y\,dy = -x\,dx$

$\int y\,dy = -\int x\,dx$

$\frac{y^2}{2} + C_1 = -\frac{x^2}{2} + C_2$

$\frac{y^2}{2} = -\frac{x^2}{2} + C$

* plug in initial condition + then solve for C *

$\frac{3^2}{2} = \frac{-4^2}{2} + C$

$9 = -16 + 2C$

$\frac{25}{2} = C$

$\frac{y^2}{2} = -\frac{x^2}{2} + 25/2$

$y = \pm\sqrt{25 - x^2}$

pos b/c initial condition of (4, 3)

$\boxed{y = \sqrt{25 - x^2}}$

domain $(-5, 5)$

cool check:

$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-1/2}(-2x)$

$= \frac{-x}{\sqrt{25 - x^2}}$

$= -x/y$

b. $\frac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$

$\int \frac{1}{\cos^2 y}\,dy = \int dx$

$\int \sec^2 y\,dy = x + C$

$\tan y = x + C$

$\tan(0) = 0 + C$

$0 = C$

$\tan y = x$

$\boxed{y = \tan^{-1}(x)}$

c. $\frac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$

$\frac{dy}{dx} = \frac{e^x}{e^y}$

$\int e^y\, dy = \int e^x\, dx$

$e^y = e^x + C$

$e^2 = e^0 + C$

$e^2 - 1 = C$

$e^y = e^x + e^2 - 1$

$\boxed{y = \ln(e^x + e^2 - 1)}$

d. $\frac{dy}{dx} = 2xy$ and $y = 3$ when $x = 0$

$\int \frac{1}{y} = \int 2x\, dx$

$\ln|y| = x^2 + C$

$\ln 3 = 0^2 + C$

$\ln 3 = C$

$\ln|y| = x^2 + \ln 3$

$y = e^{x^2 + \ln 3}$ ∗C always pos

$y = e^{x^2} e^{\ln 3}$

$\boxed{y = 3e^{x^2}}$

e. $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$

$\int \frac{1}{4\sqrt{y}}\, dy = \int \frac{\ln x}{x}\, dx$ $u = \ln x$, $du = \frac{1}{x}dx$

$\frac{1}{4}(2y^{1/2}) = \int u\, du$

$\frac{1}{2}y^{1/2} = \frac{1}{2}u^2 + C$

$\frac{1}{2}\sqrt{y} = \frac{1}{2}(\ln x)^2 + C$

$\sqrt{1} = (\ln e)^2 + 2C$

$1 = 1^2 + 2C$

$0 = C$

$y^{1/2} = (\ln x)^2$

domain $(0, \infty)$

$\boxed{y = (\ln x)^4}$

f. $\frac{dy}{dx} = (xy)^2$ and $y = 1$ when $x = 1$

$\frac{dy}{dx} = x^2y^2$

$\int y^{-2}\, dy = \int x^2\, dx$

$-y^{-1} = \frac{x^3}{3} + C$

$-1^{-1} = \frac{1}{3} + C$

$-\frac{4}{3} = C$

$-y^{-1} = \frac{x^3}{3} - \frac{4}{3}$

$y^{-1} = \frac{4 - x^3}{3}$

$\boxed{y = \frac{3}{4 - x^3}}$

In many applications, the rate of change of a variable is proportional to the value of y. In these problems, k represents the proportionality constant. Two general types of variation are direct and indirect variation. If y is a function of time t, the proportions can be written as follows:

Direct: $\frac{dy}{dt} = ky$

$\int \frac{1}{y}\,dy = \int k\,dt$

$\ln y = kt + c$

$y = e^{kt+c}$

★ $y = Ce^{kt}$ ★

C is original amount

Indirect: $\frac{dy}{dt} = \frac{k}{y}$

$\int y\,dy = \int k\,dt$

$\frac{1}{2}y^2 = kt + c$

$y^2 = 2kt + C$

$y = \sqrt{2kt + C}$

1. The rate of change of y is proportional to y. When $t = 0$, $y = 2$. What is the value of y when $t = 3$?

when $t = 2$, $y = 4$

$\frac{dy}{dt} = ky$

$y = Ce^{kt}$

$2 = Ce^{k(0)}$

$2 = C$

$y = 2e^{kt}$

$4 = 2e^{k(2)}$

$2 = e^{2k}$

$\ln 2 = 2k$

$\frac{1}{2}\ln 2 = k$

$y = 2e^{(\frac{1}{2}\ln 2)t}$

$y = 2e^{(\frac{1}{2}\ln 2)(3)}$

$\boxed{y \approx 5.657}$

2. The half-life of Plutonium-239 is 24,100 years. Suppose that 10 grams of Plutonium-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

$y = Ce^{kt}$

$y = 10e^{kt}$

Find k:

$5 = 10e^{k(24,100)}$

$\frac{1}{2} = e^{24\,100k}$

$\ln 1/2 = 24,100k$

$\frac{\ln(1/2)}{24,100} = k$

$y = 10e^{\left(\frac{\ln 1/2}{24,100}\right)t}$

$1 = 10e^{\left(\frac{\ln 1/2}{24,100}\right)t}$

$\frac{1}{10} = e^{\frac{\ln(1/2)}{24,100}t}$

$\ln(1/10) = \frac{\ln(1/2)}{24,100}t$

$\frac{\ln(1/10)}{\ln(1/2)/24,100} = t$

$\boxed{80,058.467 \text{ years} = t}$

3. Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

$y = Ce^{kt}$

$100 = Ce^{2k}$

$300 = Ce^{4k}$

$C = \frac{100}{e^{2k}}$

$300 = \frac{100}{e^{2k}} e^{4k}$

$3 = e^{2k}$

$\frac{\ln 3}{2} = k$

$C = \frac{100}{e^{2(\ln 3/2)}}$

$C \approx 33.\overline{3}$

$\boxed{\text{33 flies in the original population}}$

4. **Newton's Law of Cooling**
Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. The object cools from 100° to 90° in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°? Newton's Law of Cooling: $\frac{dT}{dt} = k(T - M)$ where M is the surrounding temperature and T is the temperature of the object.

$M = 60°$
$t = 0 \quad T = 100°$
$t = 10 \quad T = 90°$
$t = ? \quad T = 80°$

$\frac{dT}{dt} = k(T - M)$

$\frac{dT}{dt} = k(T - 60)$

$\frac{1}{T-60}\, dT = k\,dt$

$\int \frac{1}{T-60}\, dT = \int k\,dt$

$\ln|T - 60| = kt + C$

$T - 60 = Ce^{kt}$

$T = 60 + Ce^{kt}$

initial condition solve for c:

$100 = 60 + Ce^{k(0)}$

$40 = C$

$T = 60 + 40e^{kt}$

$90 = 60 + 40e^{k(10)}$

$\frac{3}{4} = e^{10k}$

$\frac{\ln(3/4)}{10} = k$

$T = 60 + 40e^{\frac{\ln(3/4)}{10}t}$

$80 = 60 + 40e^{\frac{\ln(3/4)}{10}t}$

$\frac{2}{4} = e^{\frac{\ln(3/4)}{10}t}$

$\ln \frac{1}{2} = \frac{\ln(3/4)}{10} t$

$\frac{10 \ln \frac{1}{2}}{\ln 3/4} = t$

$t = 24.094$

$\boxed{\text{It take 14.094 minutes to cool from 90° to 80°}}$