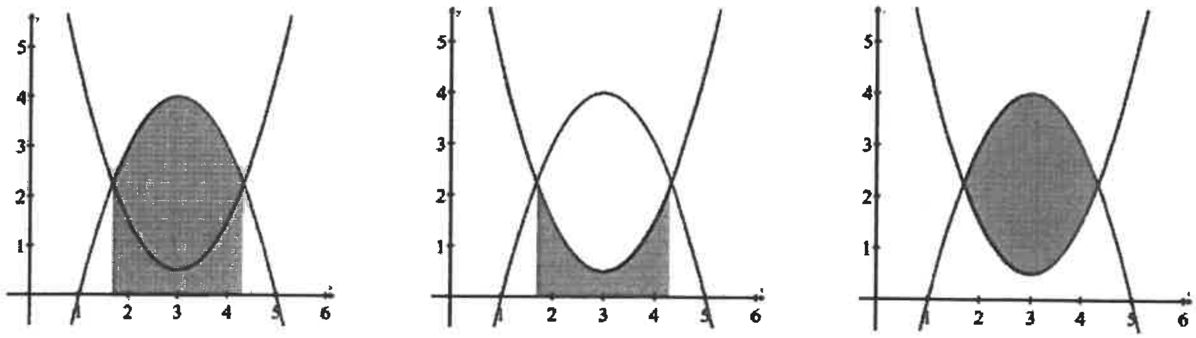
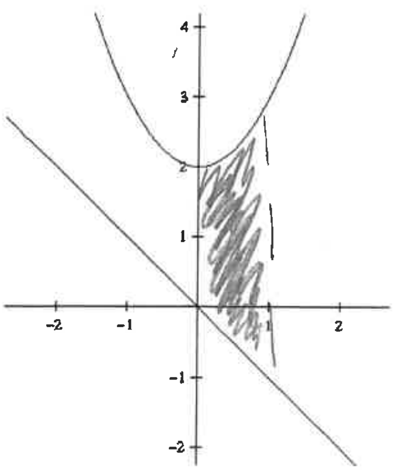
We can extend the idea of definite integrals finding the area of a region *under* a curve to the area of a region *between* two curves. If two functions are both continuous on an interval [a, b], then the region between the curves can be found by **subtracting the area of the upper region and the area of the lower region.**



Area of upper function (-) Area of lower function = Area between the functions

f(x) g(x) $\int_a^b (f(x)-g(x))dx$

1. Find the area of the region bounded by $y = x^2 + 2$, $y = -x$, $x = 0$, $x = 1$.



$$\int_0^1 [(x^2+2) - (-x)]\,dx$$

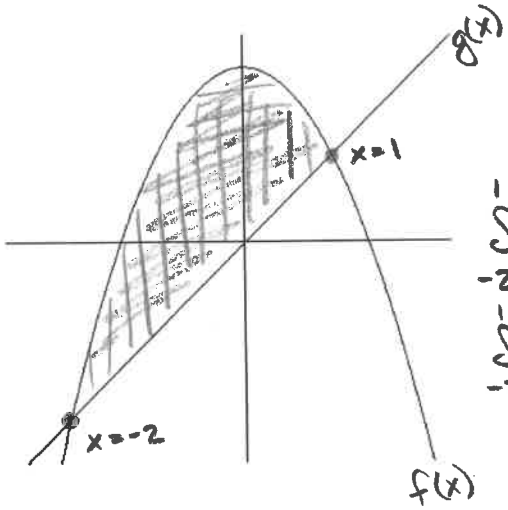
$$= \int_0^1 (x^2 + x + 2)\,dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + 2 - (0)$$

$$= \frac{17}{6}$$

2. Find the area of the region bounded by $f(x) = 2 - x^2$, $g(x) = x$



$$\int_{-2}^{1} [f(x) - g(x)]\,dx$$

$$\int_{-2}^{1} [(2-x^2) - (x)]\,dx$$

$$\int_{-2}^{1} (-x^2 - x + 2)\,dx$$

$$= 9/2$$

intersection

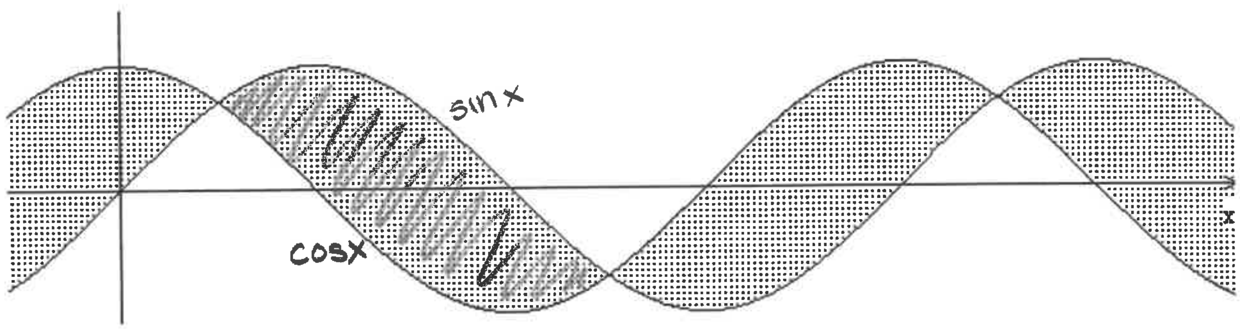
$2 - x^2 = x$

$0 = x^2 + x - 2$

$0 = (x+2)(x-1)$

$x = -2,\ 1$

3. The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of each one of these regions.



intersection points

$\sin x = \cos x$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$$\int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\,dx$$

$$= -\cos x - \sin x \Big|_{\pi/4}^{5\pi/4}$$

$$= -\cos 5\pi/4 - \sin 5\pi/4 - (-\cos \pi/4 - \sin \pi/4)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= \boxed{2\sqrt{2}}$$

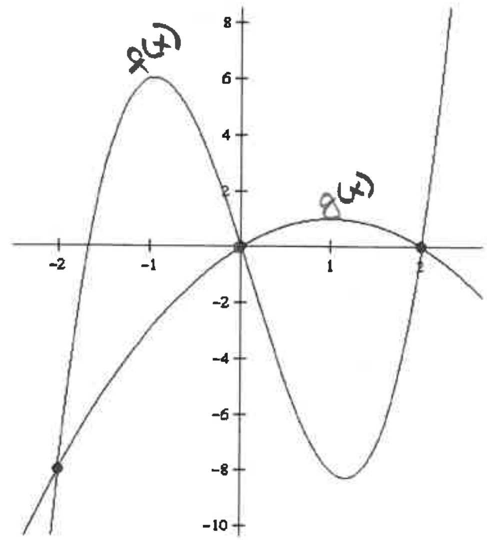
AB Calculus
8.2 Areas in the Plane

5. Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$

$$\int_{-2}^{0} (f(x) - g(x))\,dx + \int_{0}^{2} (g(x) - f(x))\,dx$$

$$\int_{-2}^{0} ((3x^3 - x^2 - 10x) - (-x^2 + 2x))\,dx +$$

$$\int_{0}^{2} ((-x^2 + 2x) - (3x^3 - x^2 - 10x))\,dx$$



$$\int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{2} (-3x^3 + 12x)\,dx$$

* functions cross & one function above from (-2,0) and then other above (0,2)

$$\frac{3}{4}x^4 - 6x^2 \Big|_{-2}^{0} + \left(-\frac{3}{4}x^4 + 6x^2\right)\Big|_{0}^{2}$$

$$0 - \left(\frac{3}{4}(-2)^4 - 6(-2)^2\right) + \left(-\frac{3}{4}(2)^4 + 6(2)^2\right) - 0$$

$$-12 + 24 - 12 + 24$$

$$24$$

If a function is written in terms of y, it is often most convenient to represent the area as an integral with respect to y. If that is the case, instead of upper – lower you will have

right - left.

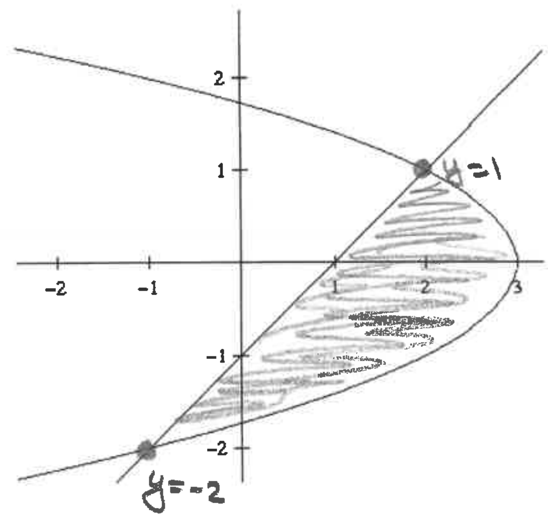
6. Find the area of the region bounded by the graphs of $f(y) = 3 - y^2$ and $g(y) = y + 1$

*y-values for bounds

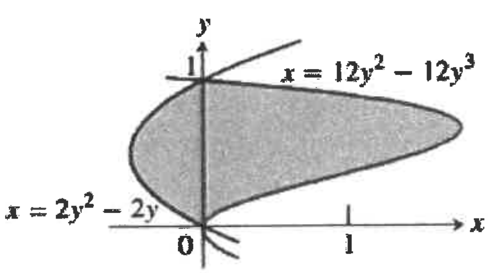
$$\int_{-2}^{1} [(3-y^2) - (y+1)]\,dy$$

$$\int_{-2}^{1} (-y^2 - y + 2)\,dy$$

$$= 4.5$$



7. Find the area of the shaded region analytically



$$\int_{0}^{1} [(12y^2 - 12y^3) - (2y^2 - 2y)]\,dy$$

$$\int_{0}^{1} (-12y^3 + 10y^2 + 2y)\,dy$$

$$-3y^4 + \frac{10}{3}y^3 + y^2 \Big|_0^1$$

$$-3 + \frac{10}{3} + 1 \quad - (0)$$

$$\frac{4}{3}$$

8. Find the area of the regions enclosed by $x = y^2$ and $x = y + 2$.

$$\int_{-1}^{2} \left((y+2) - (y^2)\right) dy$$

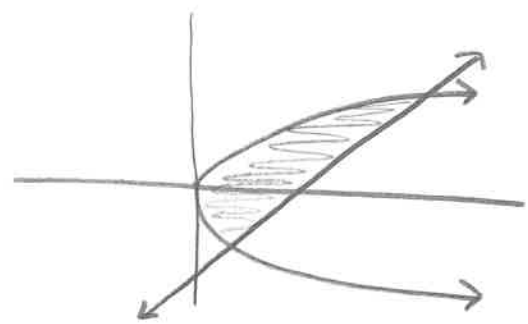
$$= 4.5$$

$y^2 = y + 2$

$0 = y^2 - y - 2$

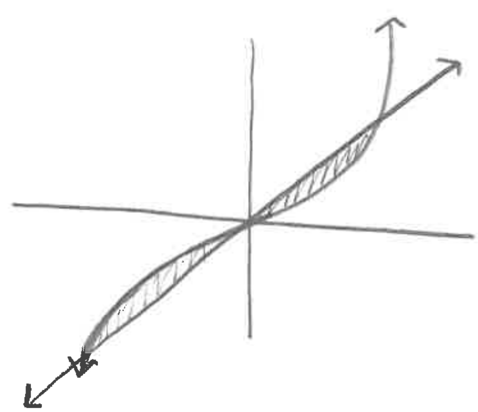
$0 = (y-2)(y+1)$

$y = 2, -1$



9. Find the area of the propeller shaped region enclosed by $y - x^3 = 0$ and $x - y = 0$

$y = x^3$ $\quad$ $y = x$



$$\int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

or

$$2\int_{0}^{1} (x - x^3)\,dx$$

$$= \frac{1}{2}$$

$x^3 = x$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x-1)(x+1) = 0$

$x = 0, 1, -1$