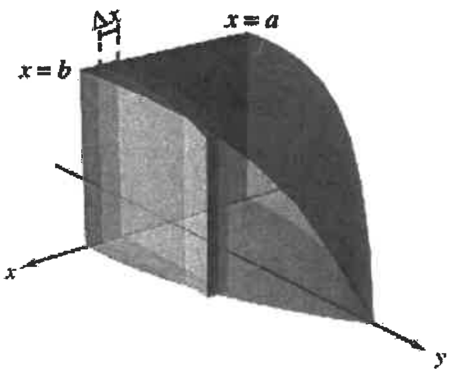
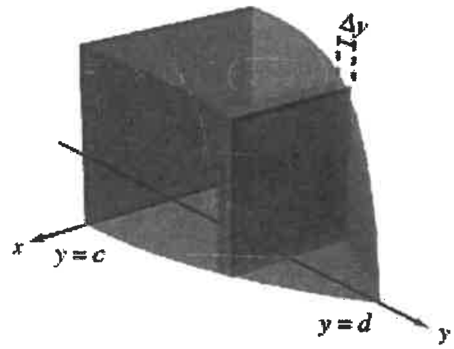
Volume by Cross Sections

Two-dimensional cross sections can be stacked on top of (or next to) each other to create a 3D figure. We use the area between two curves to represent the base of the cross sectional shape.





Volume of a Solid:

Cross Sections taken **PERPENDICULAR** to the **X-AXIS**
The volume of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b.

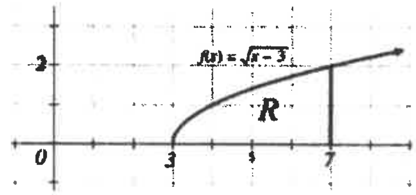
$$V = \int_a^b A(x)dx.$$

Cross Sections taken **PERPENDICULAR** to the **Y-AXIS**
The volume of a solid of known integrable cross section area $A(y)$ from $y = c$ to $y = d$ is the integral of A from c to d

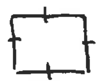
$$V = \int_C^b A(y)dy.$$

Where $A(x)$ and $A(y)$ are known geometric formulas for the area of a shape.
Example: square cross-sections → $A = s^2$, semi-circle cross sections → $A = \frac{1}{2}\pi r^2$, equilateral triangle cross-sections → $A = \frac{\sqrt{3}}{4}s^2$

1. Let R be the region in the first quadrant under the graph of $y = \sqrt{x-3}$ for $3 \le x \le 7$. Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis (vertical cross sections) are:



a. Squares



$\sqrt{x-3}$

$s = \sqrt{x-3}$

$A = (\sqrt{x-3})^2$

$$V = \int_3^7 (\sqrt{x-3})^2\,dx = 8$$

b. Equilateral Triangles

$S = \sqrt{x-3}$

$A = \frac{\sqrt{3}}{4}(\sqrt{x-3})^2$

$$V = \int_3^7 \frac{\sqrt{3}}{4}(\sqrt{x-3})^2\,dx = 2\sqrt{3}$$

c. Semicircles

$\sqrt{x-3}$

$d = \sqrt{x-3}$

$r = \frac{1}{2}\sqrt{x-3}$

$A = \frac{1}{2}\pi\left(\frac{1}{2}\sqrt{x-3}\right)^2$

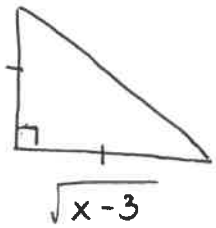
$$V = \int_3^7 \frac{1}{2}\pi\left(\frac{1}{2}(\sqrt{x-3})\right)^2 dx = \pi$$

d. Rectangle with height of 5

$$V = \int_3^7 5\sqrt{x-3}$$

$$= \frac{80}{3}$$

e. Isosceles Right Triangle where the leg is the base.



$$A = \frac{1}{2}\left(\sqrt{x-3}\right)\left(\sqrt{x-3}\right)$$

$$V = \int_3^7 \frac{1}{2}\left(\sqrt{x-3}\right)^2 dx$$

$$= 4$$

2. Let R be the region in the first quadrant under the graph of $y = \sqrt{x-3}$ for $3 \le x \le 7$. Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the y-axis (horizontal cross sections) are:

   a. Squares

$$y = \sqrt{x-3}$$

$$y^2 + 3 = x$$

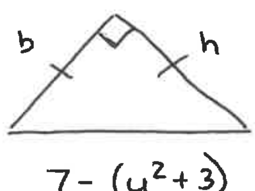
$$S = 7-(y^2+3)$$

$$S = (7-(y^2+3))^2$$

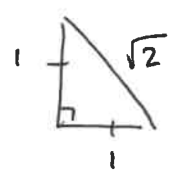
$$V = \int_0^2 (7-(y^2+3))^2 dy$$

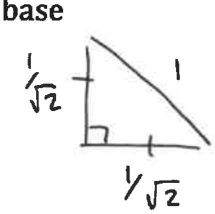
$$= 256/15$$

   b. Isosceles Right Triangles where the hypotenuse is the base



$$A = \frac{1}{2}bh$$





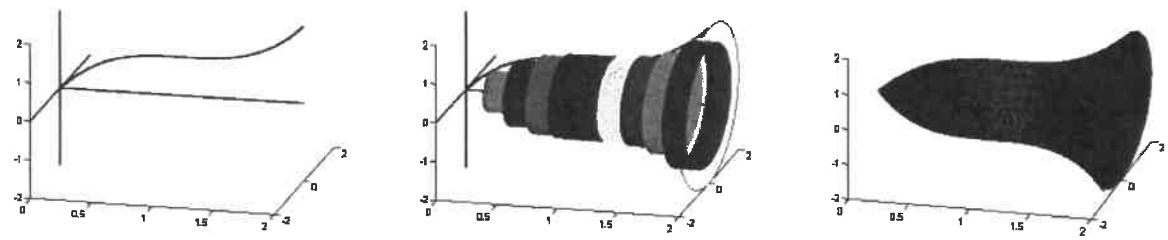
$$A = \frac{1}{2}\left(1/\sqrt{2}\,(7-(y^2+3))\right)^2$$

$$V = \int_0^2 \frac{1}{2}\left(1/\sqrt{2}\,(7-(y^2+3))\right)^2 dy$$

$$= \frac{64}{15}$$

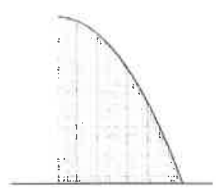
Volume by Disk Method

Create a 3-dimensional region is by rotating a function around a line. The rotation creates circular cross-sections that combine to create the volume. The resulting solid is called the **solid of revolution**, and the line that it revolved around is called the **axis of revolution**. The area of each circle is $A = \pi r^2$, where r is distance from the function to the axis of revolution.

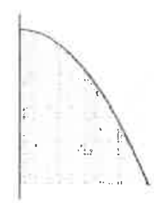


To find the volume of a solid of revolution with the disk method, use one of the following formulas:

Horizontal Axis of Revolution

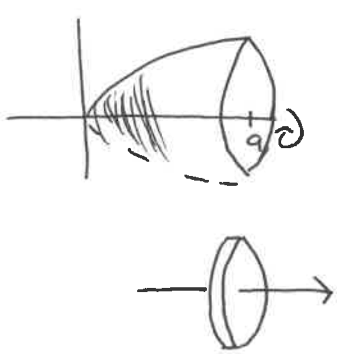


Vertical Axis of Revolution



2. **Using the Disk Method x-axis**

Find the volume of the region enclosed by $f(x) = \sqrt{x}$ and the x-axis from $[0, 9]$ rotated about the x-axis.



$r = \sqrt{x}$

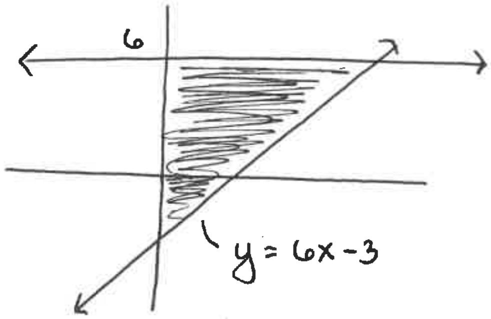
$A = \pi (\sqrt{x})^2$

$V = \int_0^9 \pi (\sqrt{x})^2 \, dx$

$= \frac{81\pi}{2}$

3. Using the Disk Method y-axis

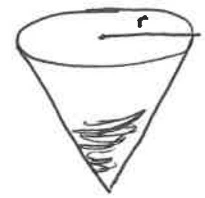
Find the volume of the solid generated when the area bounded by lines $y = 6x - 3$, $x = 0$, and $y = 6$ is revolved around the y –axis.



$$\frac{y+3}{6} = x$$

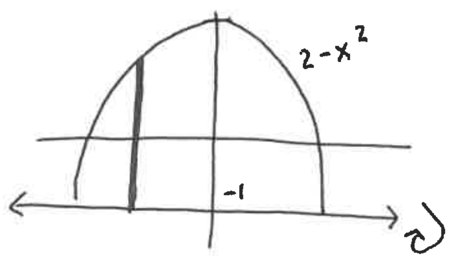
$$V = \int_{-3}^{6} \pi \left(\frac{y+3}{6}\right)^2 dy$$

$$= \frac{27\pi}{4}$$



4. Revolving About a Line That is Not a Coordinate Axis

Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = -1$ about the line $y = -1$.



$$r = 2 - x^2 - (-1)$$
$$= 3 - x^2$$

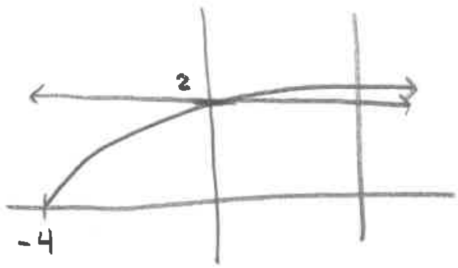
$$2 - x^2 = -1$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi (3 - x^2)^2 \, dx$$

$$= \frac{48\pi\sqrt{3}}{5}$$

Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x+4}$, $y = 2$, and $x = 2$ about the line $x = 2$.

$y^2 - 4 = x$



$r = 2 - (y^2 - 4)$

$= 2 - y^2 + 4$

$= 6 - y^2$

$$V = \int_0^2 \pi (6 - y^2)^2 \, dy$$

$$= \frac{232\pi}{5}$$

Volume by Washer Method

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative **washer**. The washer is formed by revolving a rectangle about an axis.

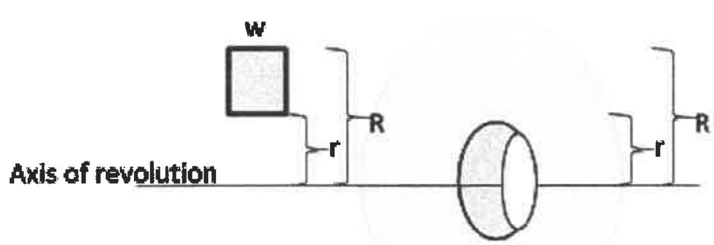
Washer Method:

If r is the inner radius and R is the outer radius, then

$$V = \pi \int_a^b [R(x)^2 - r(x)^2] dx$$

or,

$$V = \pi \int_a^b R(x)^2 dx - \pi \int_a^b r(x)^2 dx$$



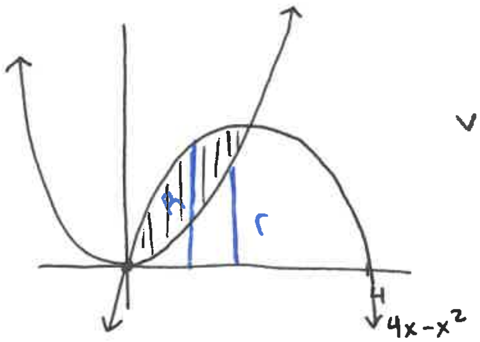
8. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = 4x - x^2$ and $f(x) = x^2$ from $[0, 2]$ about the...

a) x-axis.

$R = 4x - x^2$ $\quad r = x^2$

$$V = \int_0^2 \left(\pi(4x-x^2)^2 - \pi(x^2)^2\right)dx$$

$$= \frac{32\pi}{3}$$



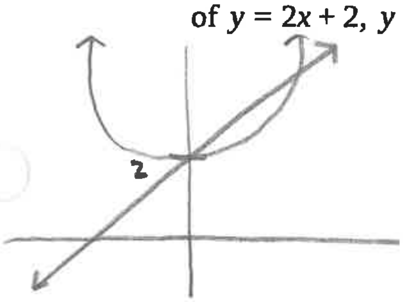
b) the line $y = 5$

$R = 5 - (x^2)$ $\quad r = 5 - (4x - x^2)$

$$V = \pi \int_0^2 (5 - x^2)^2 - (5 - (4x - x^2))^2 dx$$

$$= 16\pi$$

9. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = 2x + 2$, $y = x^2 + 2$ about the x-axis.



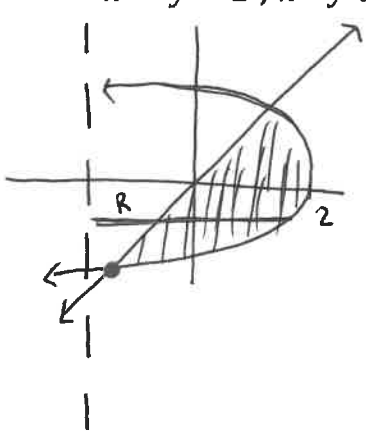
$R = 2x + 2$ $\quad r = x^2 + 2$

$2x + 2 = x^2 + 2$
$0 = x^2 - 2x$
$x = 0, 2$

$$V = \int_0^2 \left(\pi(2x+2)^2 - \pi(x^2+2)^2\right)dx$$

$$= \frac{48\pi}{5}$$

10. Find the volume of the solid generated by revolving the region bounded by $x = -y^2 + 2$, $x = y$ about the line $x = -3$.



$R = -y^2 + 2 - (-3)$
$= -y^2 + 5$

$r = y - (-3)$
$= y + 3$

$-y^2 + 2 = y$
$0 = y^2 + y - 2$
$0 = (y+2)(y-1)$
$y = -2, 1$

$$V = \int_{-2}^{1} \left[\pi(-y^2+5)^2 - \pi(y+3)^2\right]dy$$

$$= \frac{153\pi}{5}$$