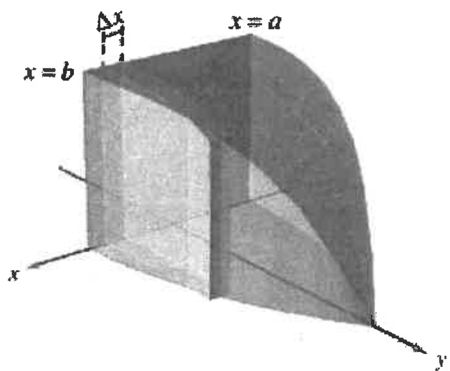
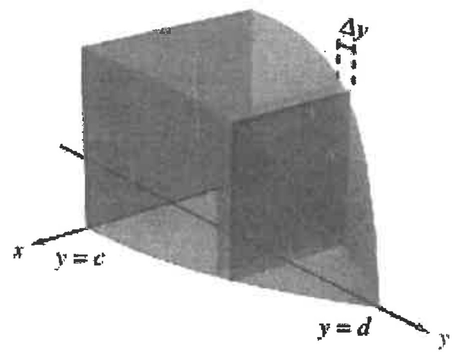
Volume by Cross Sections

Two-dimensional cross sections can be stacked on top of (or next to) each other to create a 3D figure. We use the area between two curves to represent the base of the cross sectional shape.





Volume of a Solid:

Cross Sections taken **PERPENDICULAR** to the **X-AXIS**
The volume of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b.

$$V = \int_a^b A(x)dx.$$

Area · dx = Volume

Cross Sections taken **PERPENDICULAR** to the **Y-AXIS**
The volume of a solid of known integrable cross section area $A(y)$ from $y = c$ to $y = d$ is the integral of A from c to d

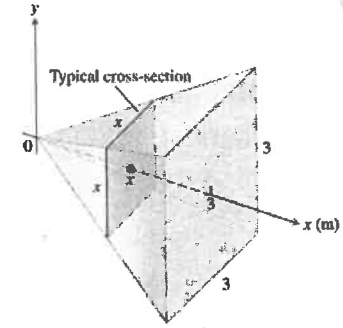
$$V = \int_C^b A(y)dy.$$

* everything in terms of y including bounds!

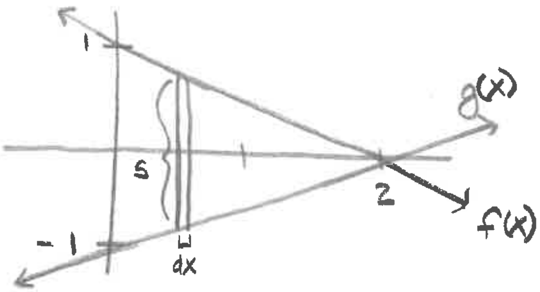
Where $A(x)$ and $A(y)$ are known geometric formulas for the area of a shape.
Example: square cross-sections → $A = s^2$, semi-circle cross sections → $A = \frac{1}{2}\pi r^2$, equilateral triangle cross-sections → $A = \frac{\sqrt{3}}{4}s^2$

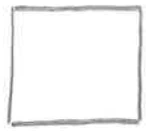
1. Use the following region for each cross-section:

Find the volume of a triangular shaped solid whose base is the region bounded by the lines $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$ and $x = 0$ using....



a) Square cross sections perpendicular to the x-axis





one cross section

s

s

$s = f(x) - g(x)$
$= (1 - x/2) - (-1 + x/2)$
$= 2 - x$

$A(x) = s^2$
$= (2-x)^2$

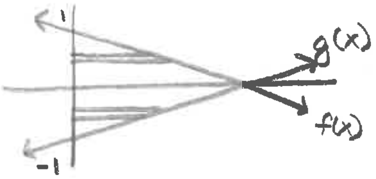
$V = \int_0^2 (2-x)^2 \, dx$

$= \left. \frac{-(2-x)^3}{3} \right|_0^2$

$= -\frac{(2-2)^3}{3} + \frac{(2-0)^3}{3}$

$\boxed{= \frac{8}{3}}$

b) Square cross sections perpendicular to the y-axis

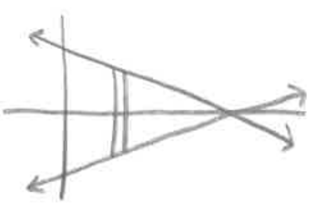


$V = \int_{-1}^{0} (2+2y)^2 \, dy + \int_0^1 (2-2y) \, dy$

$= \boxed{\frac{8}{3}}$

s: $y = 1 - x/2 \Rightarrow x = 2 - 2y$
and
$y = -1 + x/2 \Rightarrow x = 2 + 2y$

c) Semi-circles perpendicular to the x-axis



$A(x) = \frac{1}{2}\pi(1 - x/2)^2$

$V = \int_0^2 \frac{1}{2}\pi(1 - x/2)^2 \, dx$

$= \frac{\pi}{2} \int_0^2 (1 - x/2)^2$

r

$A = \frac{1}{2}\pi r^2$

$r = \frac{1}{2}(f(x) - g(x))$
or
$r = 1 - x/2 = f(x)$

$u = 1 - x/2$
$du = -1/2 \, dx$
$-2du = dx$

$= -2 \cdot \frac{\pi}{2} \int_1^0 u^2 \, du$

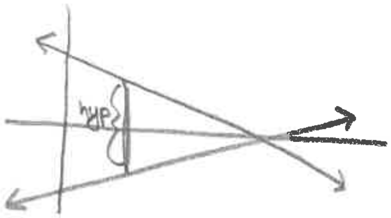
$= \pi \int_0^1 u^2 \, du$

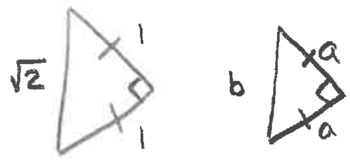
$= \left. \pi \frac{u^3}{3} \right|_0^1$

$= \frac{\pi}{3} - 0$

$\boxed{= \frac{\pi}{3}}$

d) Isosceles right triangles where the hypotenuse is bound by the region and perpendicular to the x-axis.





hypotenuse $= f(x) - g(x)$

$= 2 - x$

$b^2 = a^2 + a^2$

$b^2 = 2a^2$

$\frac{b^2}{2} = a^2$

$\sqrt{\frac{b^2}{2}} = a$

$\frac{1}{\sqrt{2}} b = a$

$\Rightarrow \frac{1}{\sqrt{2}}(2-x) = a$

$A = \frac{1}{2}$ base $\cdot$ height

$= \frac{1}{2} a \cdot a$

$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}(2-x)\right)^2$

$V = \int_0^2 \frac{1}{2}\left(\frac{1}{\sqrt{2}}(2-x)\right)^2 dx$

$= \frac{1}{4}\int_0^2 (2-x)^2 dx$

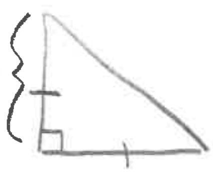
$= -\frac{1}{12}(2-x)^3 \Big|_0^2$

$= -\frac{1}{12}(2-2)^3 + \frac{1}{12}(2-0)^3$

$\boxed{= \frac{2}{3}}$

e) Isosceles right triangles whose leg is bounded by the region and perpendicular to the x-axis

$f(x) - g(x)$



$f(x) - g(x) = 2 - x$

$2 - x =$ height $=$ base

$A = \frac{1}{2}(2-x)(2-x)$

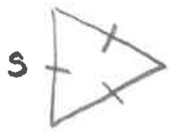
$V = \int_0^2 \frac{1}{2}(2-x)^2 dx$

$= \frac{-1}{2} \frac{(2-x)^3}{3} \Big|_0^2$

$= -\frac{1}{6}(2-2)^3 + \frac{1}{6}(2-0)^3$

$\boxed{= \frac{4}{3}}$

f) Equilateral triangle perpendicular to the x-axis.



$s = f(x) - g(x)$

$s = 2 - x$

$A = \frac{\sqrt{3}}{4}(2-x)^2$

$V = \int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,dx$

$= -\frac{\sqrt{3}}{4}\frac{(2-x)^3}{3}\Big|_0^2$

$= -\frac{\sqrt{3}}{12}(2-2)^3 + \frac{\sqrt{3}}{12}(2-0)^3$

$\boxed{= \frac{2\sqrt{3}}{3}}$