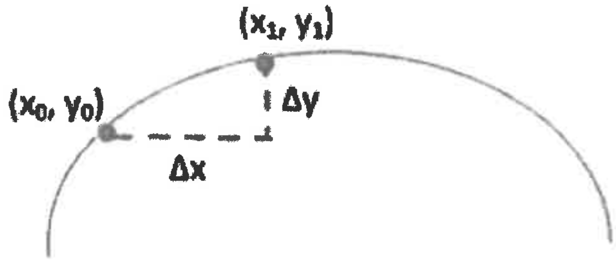
BC Calculus

8.4 Length of Curves

Arc Length:

The length of a curve is found by summing small segments using distance formula



$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2 \left(\frac{\Delta x}{\Delta x}\right)^2}$$ multiply by "1"

$$d = \sqrt{(\Delta x)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2 (\Delta x)^2}$$ pull Δx in with Δy

$$d = \sqrt{(\Delta x)^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$ factor out $(\Delta x)^2$

$$d = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,(\Delta x)$$ $\sqrt{(\Delta x)^2} = \Delta x$ ← take out of square root

$$d = \sqrt{1 + f'(x)^2}(\Delta x)$$ $\frac{\Delta y}{\Delta x}$ = slope so replace w/ $f'(x)$

$$length = \int_a^b \sqrt{1 + f'(x)^2}dx$$ if Δx is infinitely small, take an integral to add up all the little pieces

Smoothness:

A function with a continuous first derivative is **smooth** and its graph is a **smooth curve.**

DEFINITION Arc Length: Length of a Smooth Curve

If a smooth curve begins at (a, c) and ends at (b, d), $a < b$, $c < d$, then the **length (arc length) of the curve is**

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{if } y \text{ is a smooth function of } x \text{ on } [a, b];$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad \text{if } x \text{ is a smooth function of } y \text{ on } [c, d].$$

1. Find the arc length of $f(x) = x^2$ on the interval of $[-1, 1]$.

$f(x) = x^2$

$f'(x) = 2x$

$$\text{length} = \int_{-1}^{1} \sqrt{1 + (2x)^2}\, dx$$

$$= 2.958$$

2. Find the arc length of $f(x) = \frac{3}{2}x^{\frac{2}{3}}$ on the interval of $[1, 8]$.

$f(x) = \frac{3}{2}x^{2/3}$

$f'(x) = \frac{3}{2} \cdot \frac{2}{3} x^{-1/3}$

$= x^{-1/3}$

$$\text{length} = \int_{1}^{8} \sqrt{1 + (x^{-1/3})^2}\, dx$$

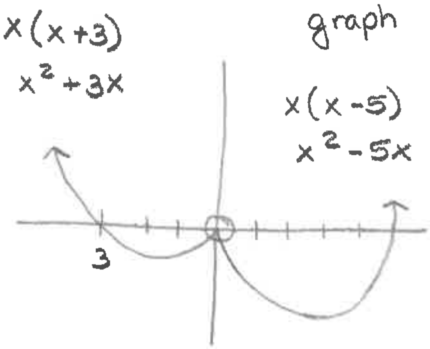
$$= 8.352$$

Getting around a corner:

3. Find the length of the curve $y = x^2 - 4|x| - x$ from $x = -4$ to $x = 4$.

★ absolute value should always be cause for alarm

graph to see



$$x^2 - 4|x| - x = \begin{cases} x^2 + 3x, & x < 0 \\ x^2 - 5x, & x \geq 0 \end{cases}$$

corner at $x=0$

neither $\frac{dy}{dx}$ nor $\frac{dx}{dy}$ exists

$\Rightarrow$ find length of 2 smooth pieces

$$L = \int_{-4}^{0} \sqrt{1 + (2x+3)^2}\,dx + \int_{0}^{4} \sqrt{1 + (2x-5)^2}\,dx$$

$= 19.56$

4. Find the length of the curve $x = \int_0^y \sqrt{\sec^2 t - 1}\,dt$, $-\frac{\pi}{3} \leq y \leq \frac{\pi}{4}$

$$\frac{dx}{dy} = \sqrt{\sec^2 y - 1}\quad dy$$

$$L = \int_{-\pi/3}^{\pi/4} \sqrt{1 + \left(\sqrt{\sec^2 y - 1}\right)^2}\,dy$$

$= 2.198$

5) Find the length of the curve $y = x^{1/3}$ on the interval of $[-8, 8]$

$\frac{dy}{dx} = \frac{1}{3} x^{-2/3}$

★ vertical tangent at $x = 0$ (slope = ∞)

if we change to x as a function of y the tangent at the origin will be horizontal and the derivative will be 0

$x = y^3$

$\frac{dx}{dy} = 3y^2$

$L = \int_{-2}^{2} \sqrt{1 + (3y^2)^2}\, dy$

$= 17.26$

★ change bounds for new function

$y = \sqrt[3]{-8} = -2$

$y = \sqrt[3]{8} = 2$

$[-2, 2]$