$\lim_{x \to \infty} \frac{f(x)}{g(x)}$

1) $= 0$ then g larger

Faster, Slower, Same-Rate Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f grows faster than g (and g grows slower than f) as $x \to \infty$ if
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty, \text{ or equivalently, if } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

2. f and g grow at the same rate as $x \to \infty$ if
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0$$
L finite and not 0

$= \infty$ then f larger

1. Show that e^x grows faster than the given function:

a. x^{20}

$\lim_{x \to \infty} \frac{e^x}{x^{20}}$ take L'hopitals rule 20 times

$\lim_{x \to \infty} \frac{e^x}{20!} = \infty$

e^x grows faster than x^{20}

b. $\left(\frac{5}{2}\right)^x$

$e > 5/2$

$\lim_{x \to \infty} \frac{e^x}{(5/2)^x} = \infty$

e^x grows faster than $\left(\frac{5}{2}\right)^x$

2. Show that $\ln x$ grows slower than the given function:

a. $\sqrt{x}$

$\lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} \quad \frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{\frac{1}{2}x^{-1/2}}{1/x}$

$= \lim_{x \to \infty} \frac{x}{2\sqrt{x}}$

$= \lim_{x \to \infty} \frac{\sqrt{x}}{2} = \infty$

$\ln x$ grows slower than $\sqrt{x}$

b. x^3

$\lim_{x \to \infty} \frac{x^3}{\ln x} \quad \frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{3x^2}{1/x}$

$= \lim_{x \to \infty} 3x^3$

$= \infty$

$\ln x$ grows slower than x^3

3. Show that x^2 grows at the same rate as the given function:

a. $\sqrt{x^4 + 5x}$

$$\lim_{x\to\infty} \frac{\sqrt{x^4+5x}}{x^2}$$

$$= \lim_{x\to\infty} \sqrt{\frac{x^4+5x}{x^4}}$$

$$= \lim_{x\to\infty} \sqrt{1+\frac{5}{x^3}}$$

$$= 1$$

b. $x^2 + \sin x$

$$\lim_{x\to\infty} \frac{x^2+\sin x}{x^2}$$

$$= \lim_{x\to\infty} 1 + \frac{\sin x}{x^2}$$

$$= \lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{\sin x}{x^2}$$

$$= 1 + 0$$

$$= 1$$

4. Show that the two functions, e^{x+1}, e^x grow at the same rate

$$\lim_{x\to\infty} \frac{e^{x+1}}{e^x}$$

$$= \lim_{x\to\infty} \frac{\cancel{e^x} e^1}{e^x}$$

$$= \lim_{x\to\infty} e$$

$$= e$$

5. Determine whether the functions grows faster, at the same rate, or slower than e^x as $x \to \infty$

a. 4^x

$4 > e$

$$\lim_{x\to\infty} \frac{e^x}{4^x}$$

$$= 0$$

4^x grows faster than e^x

b. xe^x

$$\lim_{x\to\infty} \frac{e^x}{xe^x}$$

$$= \lim_{x\to\infty} \frac{1}{x}$$

$$= 0$$

xe^x grows faster than e^x

6. Determine whether the functions grows faster, at the same rate, or slower than x^2 as $x \to \infty$

a. $\sqrt{x}$

$\lim_{x\to\infty} \frac{x^2}{\sqrt{x}}$

$= \lim_{x\to\infty} x^{3/2}$

$= \infty$

x^2 grows faster than $\sqrt{x}$

b. 2^x

$\lim_{x\to\infty} \frac{x^2}{2^x}$ $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2x}{2^x \ln 2}$ $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x}$

$= 0$

2^x grows faster than x^2

Transitivity of Growing Rates
If f grows at the same rate as g as $x \to \infty$ and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h as $x \to \infty$

7. Order the functions from slowest-growing to fastest-growing as $x \to \infty$

2^x, x^2, $(\ln 2)^x$, e^x

$\lim_{x\to\infty} \frac{2^x}{x^2} = \infty$ ← faster

$\lim_{x\to\infty} \frac{(\ln 2)^x}{e^x} = 0$ ← faster

* $\ln 2 < e$ *

$\lim_{x\to\infty} \frac{x^2}{(\ln 2)^x} = \infty$ ← faster

* $\ln 2 < 1$ *

$(\ln 2)^x,\ x^2,\ 2^x,\ e^x$

8. Show that the three functions grow at the same rate as $x \to \infty$

$f_1(x) = x^2,\ f_2(x) = \sqrt{x^4 + x},\ f_3(x) = \sqrt{x^4 - x^3}$

$\lim_{x\to\infty} \frac{\sqrt{x^4+x}}{x^2}$

$= \lim_{x\to\infty} \sqrt{\frac{x^4+x}{x^4}}$

$= \lim_{x\to\infty} \sqrt{1+\frac{1}{x^3}}$

$= 1$

$\lim_{x\to\infty} \frac{\sqrt{x^4-x^3}}{x^2}$

$= \lim_{x\to\infty} \sqrt{\frac{x^4-x^3}{x^4}}$

$= \lim_{x\to\infty} \sqrt{1-1/x}$

$= 1$

* Transitivity of growing rates