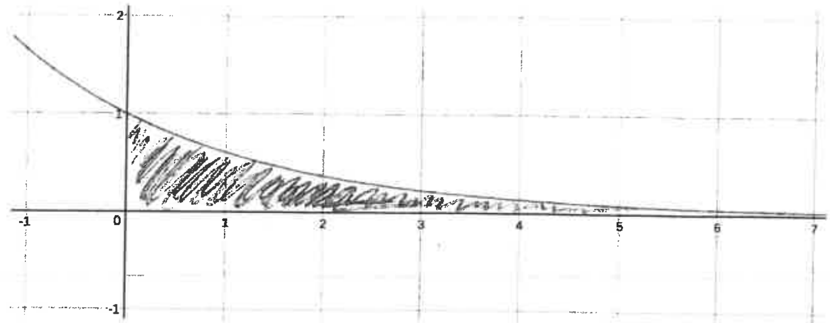
Consider $f(x) = e^{-x/2}$

What is the area under the curve from 0 to ∞?



$A(b) = \int_0^b e^{-x/2}dx \quad = -2e^{-\frac{x}{2}}\Big|_0^b \quad = -2e^{-b/2} + 2$

Find the area $A(b)$ of the portion of the region bounded on the right by $x=b$

$\lim_{n\to\infty} A(b) = \lim_{n\to\infty}(-2e^{-b/2} + 2) = 2$

Find limit as $b \to \infty$

$\int_0^\infty e^{-x/2}dx = \lim_{b\to\infty}\int_0^b e^{-x/2}dx = 2$

Improper Integrals with Infinite Integration Limits

Integrals with infinite limits of integration are **improper integrals**

1. If $f(x)$ is continuous on $[a, \infty)$, then:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

2. If $f(x)$ is continuous on $[-\infty, b]$, then:

$$\int_{-\infty}^a f(x)dx = \lim_{a\to\infty}\int_a^b f(x)dx$$

*limit finite the improper integral converges and the limit is the value of the improper integral.

*limit fails to exist the improper integral diverges.

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then:

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx,$$

where c is any real number

Limit converges if both improper integrals converge, otherwise it diverges

1. Evaluate the improper integral and determine if it converges or diverges:

a. $\int_1^{\infty} \frac{2}{x^3}dx$

$$\lim_{b\to\infty} \int_1^b 2x^{-3}\,dx$$

$$= \lim_{b\to\infty} -x^{-2}\Big|_1^b$$

$$= \lim_{b\to\infty} \left(-b^{-2} + (1)^{-2}\right)$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b^2} + 1\right)$$

$$= 0 + 1$$

$$\boxed{= 1}$$

b. $\int_1^{\infty} \frac{dx}{\sqrt[4]{x}}$

$$\lim_{b\to\infty} \int_1^b x^{-1/4}\,dx$$

$$= \lim_{b\to\infty} \frac{4}{3}x^{3/4}\Big|_1^b$$

$$= \lim_{b\to\infty} \left(\frac{4}{3}b^{3/4} - \frac{4}{3}(1)^{3/4}\right)$$

$$= \infty$$

$$\boxed{\int_1^{\infty} \frac{dx}{\sqrt[4]{x}}\,dx \text{ diverges}}$$

c. $\int_{-\infty}^{0} \frac{dx}{(x-2)^3}$

$$\lim_{a\to-\infty} \int_a^0 (x-2)^{-3}\,dx$$

* u-sub

$$= \lim_{a\to-\infty} \frac{(x-2)^{-2}}{-2}\Big|_a^0$$

$$= \lim_{a\to-\infty} \left(-\frac{1}{2}\left[(0-2)^{-2} - (a-2)^{-2}\right]\right)$$

$$= \lim_{a\to-\infty} \left(-\frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2(a-2)^2}\right)$$

$$= -\frac{1}{8} + 0 \qquad \boxed{= -\frac{1}{8}}$$

d. $\int_{-\infty}^{\infty} e^x$ let $c = 0$

$$= \lim_{a\to-\infty} \int_a^0 e^x\,dx + \lim_{b\to\infty} \int_0^b e^x\,dx$$

$$= \lim_{a\to-\infty} e^x\Big|_a^0 + \lim_{b\to\infty} e^x\Big|_0^b$$

$$= \lim_{a\to-\infty} (e^0 - e^a) + \lim_{b\to\infty} (e^b - e^0)$$

$$= (1 - 0) + (\infty - 1)$$

$$= 1 + \infty$$

$$\boxed{\int_{-\infty}^{\infty} e^x\,dx \text{ diverges}}$$

* b/c 2nd integral goes to ∞

e. $\int_1^{\infty} \frac{dx}{x}$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln|x| \Big|_1^b$$

$$= \lim_{b\to\infty} \left(\ln|b| - \cancel{\ln|1|}^{=0}\right)$$

$$= \infty$$

$$\boxed{\int_1^{\infty} \frac{1}{x}\,dx \text{ diverges}}$$

f. $\int_0^{\infty} \frac{2\,dx}{x^2+4x+3}$

$$\lim_{b\to\infty} \int_0^b \frac{2}{(x+3)(x+1)}\,dx$$

$$\frac{2}{(x+3)(x+1)} = \frac{A}{(x+3)} + \frac{B}{x+1}$$

$$2 = A(x+1) + B(x+3)$$

$x=-1$: $2 = 2B$, $1 = B$

$x=-3$: $2 = -2A$, $-1 = A$

$$\lim_{b\to\infty} \int_0^b \left(\frac{-1}{x+3} + \frac{1}{x+1}\right)dx$$

$$\lim_{b\to\infty} \left(-\ln|x+3|\Big|_0^b + \ln|x+1|\Big|_0^b\right)$$

$$\lim_{b\to\infty} \left(-\ln|b+3| + \ln|3| + \ln|b+1| - \cancel{\ln|1|}^{0}\right)$$

$$= \lim_{b\to\infty} \left(\ln|3| + \ln|b+1| - \ln|b+3|\right)$$

$$= \lim_{b\to\infty} \left(\ln|3| + \ln\left|\frac{b+1}{b+3}\right|\right)$$

$$= \lim_{b\to\infty} \left(\ln 3 + \ln\left|\frac{b(1+1/b)}{b(1+3/b)}\right|\right)$$

$$= \lim_{b\to\infty} \left(\ln 3 + \ln|1|\right)$$

* $b\to\infty$ so $1/b \to 0$, $3/b \to 0$

$$\boxed{= \ln 3}$$

g. $\int_{-\infty}^{0} \frac{2\,dx}{x^2-4x+3}$ * partial fractions

$$= \lim_{a\to-\infty} \int_a^0 \frac{2}{(x-3)(x-1)}\,dx$$

$$\frac{2}{(x-3)(x-1)} = \frac{A}{(x-3)} + \frac{B}{x-1}$$

$$2 = A(x-1) + B(x-3)$$

$x=1$: $2 = -2B$, $-1 = B$

$x=3$: $2 = 2A$, $1 = A$

$$= \lim_{a\to-\infty} \int_a^0 \left(\frac{1}{x-3} + \frac{-1}{x-1}\right)dx$$

$$= \lim_{a\to-\infty} \left(\ln|x-3|\Big|_a^0 - \ln|x-1|\Big|_a^0\right)$$

$$= \lim_{a\to-\infty} \left(\ln|-3| - \ln|a-3| - \left(\cancel{\ln|-1|}^{0} - \ln|a-1|\right)\right)$$

$$= \lim_{a\to-\infty} \left(\ln 3 - \ln|a-3| + \ln|a-1|\right)$$

$$= \lim_{a\to-\infty} \left(\ln 3 + \ln|a-1| - \ln|a-3|\right)$$

$$= \lim_{a\to-\infty} \left(\ln 3 + \ln\left|\frac{a-1}{a-3}\right|\right)$$

$$= \lim_{a\to-\infty} \left(\ln 3 + \ln\left|\frac{a(1-1/a)}{a(1-3/a)}\right|\right)$$

$$= \lim_{a\to-\infty} \left(\ln 3 + \ln\left|\frac{1-1/a}{1-3/a}\right|\right) = \ln 3 + \ln 1 \boxed{= \ln 3}$$

h. $\int_1^{\infty} xe^{-x}dx$

$= \lim_{b\to\infty} \int_1^b xe^{-x}\,dx$

*by parts LIPET

$u = x \qquad v = -e^{-x}$

$du = 1dx \qquad dv = e^{-x}dx$

$= \lim_{b\to\infty} \left(-xe^{-x}\Big|_1^b - \int_1^b -e^{-x}\,dx \right)$

$= \lim_{b\to\infty} \left(-xe^{-x}\Big|_1^b + \int_1^b e^{-x}\,dx \right)$

$= \lim_{b\to\infty} \left(-xe^{-x}\Big|_1^b - e^{-x}\Big|_1^b \right)$

$\lim_{b\to\infty} \left(-\frac{b}{e^b} + e^{-1} - (-e^{-b} - e^{-1}) \right)$

$\lim_{b\to\infty} \left(-\frac{b}{e^b} + e^{-1} + \frac{1}{e^b} + e^{-1} \right)$

$= \lim_{b\to\infty} \left(\frac{1-b}{e^b} + \frac{2}{e} \right)$

$= \lim_{b\to\infty} \left(\frac{1-b}{e^b} \right) + \lim_{b\to\infty} \frac{2}{e}$

L'hosp. $\frac{\infty}{\infty}$

$\lim_{b\to\infty} \left(\frac{-1}{e^b} \right) + \frac{2}{e}$

$\boxed{= \frac{2}{e}}$

i. $\int_0^{\infty} (x+1)e^{-x}\,dx$

$= \lim_{b\to\infty} \int_0^b (x+1)e^{-x}\,dx$

$u = x+1 \qquad v = -e^{-x}$

$du = dx \qquad dv = e^{-x}dx$

$= \lim_{b\to\infty} \left(-(x+1)e^{-x}\Big|_0^b - \int_0^b -e^{-x}\,dx \right)$

$= \lim_{b\to\infty} \left(-(x+1)e^{-x}\Big|_0^b - e^{-x}\Big|_0^b \right)$

$= \lim_{b\to\infty} \left(-(x+1)e^{-x} - e^{-x} \right)\Big|_0^b$

$= \lim_{b\to\infty} \left(e^{-x}(-x-1-1) \right)\Big|_0^b$

$= \lim_{b\to\infty} \left(e^{-x}(-x-2) \right)\Big|_0^b$

$= \lim_{b\to\infty} \left(e^{-b}(-b-2) - (e^0(-2)) \right)$

$= \lim_{b\to\infty} \left(1/e^b\,(-b-2) + 2 \right)$

$= \lim_{b\to\infty} \left(\frac{-b-2}{e^b} \right) + 2$

L'hosp $\frac{-\infty}{\infty}$

$= \lim_{b\to\infty} \left(\frac{-1}{e^b} \right) + 2$

$\boxed{= 2}$

j) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

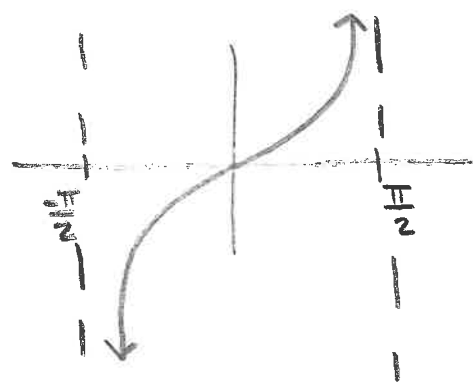
$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \tan^{-1}x \Big|_a^0 + \lim_{b\to\infty} \tan^{-1}x \Big|_0^b$$

$$= \lim_{a\to-\infty} \left(\overset{=0}{\tan^{-1}(0)} - \tan^{-1}(a)\right) + \lim_{b\to\infty} \left(\tan^{-1}(b) - \tan^{-1}(0)\right)$$

$$= \lim_{a\to-\infty} -\tan^{-1}(a) + \lim_{b\to\infty} \tan^{-1}(b)$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$



$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\boxed{= \pi}$$

Integrand with Infinite Discontinuities: integrand has a vertical asymptote!

Improper Integrals with Infinite Discontinuities

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals**

1. If $f(x)$ is continuous on $(a, b]$, then:

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

2. If $f(x)$ is continuous on $[a, b)$, then:

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$$

*limit finite the improper integral converges and the limit is the value of the improper integral.
*limit fails to exist the improper integral diverges.

3. If $f(x)$ is continuous on $[a, c) \cup (c, b]$, then:

$$\int_a^c f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Limit converges if both improper integrals converge, otherwise it diverges

2. Evaluate the following integrals and determine whether they converge or diverge:

a. $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ *vertical asymptote at $x=1$

$= \int_0^1 \frac{1}{(x-1)^{2/3}}dx + \int_1^3 \frac{1}{(x-1)^{2/3}}dx$

$= \lim_{c\to1^-} \int_0^c \frac{1}{(x-1)^{2/3}}dx + \lim_{c\to1^+} \int_c^3 \frac{1}{(x-1)^{2/3}}dx$

$= \lim_{c\to1^-} 3(x-1)^{1/3} \Big|_0^c + \lim_{c\to1^+} 3(x-1)^{1/3} \Big|_c^3$

$= \lim_{c\to1^-} 3(c-1)^{1/3} - 3(-1)^{1/3} + \lim_{c\to1^+} 3(2)^{1/3} - 3(c-1)^{1/3}$

$0 + 3 + 3\sqrt[3]{2} - 0$

$= \boxed{3 + 3\sqrt[3]{2}}$

b. $\int_1^2 \frac{dx}{(x-2)}$ vertical asymptote at 2

$= \lim_{c\to2^-} \int_1^c \frac{1}{x-2}dx$

$= \lim_{c\to2^-} \ln|x-2| \Big|_1^c$

$= \lim_{c\to2^-} \left(\ln|c-2| - \ln|-1|\right)$

$= \lim_{c\to2^-} \ln|c-2| - 0$

* $\ln|c-2| \to \ln|0| \to -\infty$

$= -\infty$

$\boxed{\text{diverges}}$

like relative rates of growth

* interesting ∫

won't have questions on it

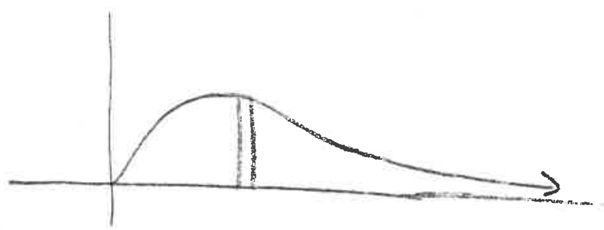
Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then:

1. $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges
2. $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges

Challenge!

3. Find the volume of the solid obtained by revolving the curve $y = xe^{-x},\ 0 \le x \le \infty$ about the x-axis.



* disks

$$\pi r^2 = \pi (xe^{-x})^2$$

$$= \pi x^2 e^{-2x}$$

$$V = \int_0^\infty \pi x^2 e^{-2x}\, dx$$

$$= \pi \lim_{b\to\infty} \int_0^b x^2 e^{-2x}\, dx$$

x^2 and derivatives	e^{-2x} and antiderivatives
x^2	(+) e^{-2x}
$2x$	$-\frac{1}{2}e^{-2x}$
2	(−) $\frac{1}{4}e^{-2x}$
0	(+) $-\frac{1}{8}e^{-2x}$

$$= \pi \lim_{b\to\infty} \left(-\frac{1}{2}x^2e^{-2x}\Big|_0^b - \frac{1}{2}xe^{-2x}\Big|_0^b - \frac{1}{4}e^{-2x}\Big|_0^b\right)$$

$$= \pi \lim_{b\to\infty} \left(e^{-2x}\left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right)\Big|_0^b\right)$$

$$= \pi \lim_{b\to\infty} \left(e^{-2b}\left(-\frac{1}{2}b^2 - \frac{1}{2}b - \frac{1}{4}\right) - e^0\left(-\frac{1}{4}\right)\right)$$

$$= \pi \lim_{b\to\infty} \left(\frac{-\frac{1}{2}b^2 - \frac{1}{2}b - \frac{1}{4}}{e^{2b}}\right) + \frac{\pi}{4}$$

L'hosp $\frac{\infty}{\infty}$

$$= \pi \lim_{b\to\infty} \left(\frac{-b - \frac{1}{2}}{\frac{1}{2}e^{2b}}\right) + \frac{\pi}{4}$$

L'hosp $\frac{-\infty}{\infty}$

$$= \pi \lim_{b\to\infty} \frac{-1}{\frac{1}{4}e^{2b}} + \frac{\pi}{4}$$

$$= 0 + \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$