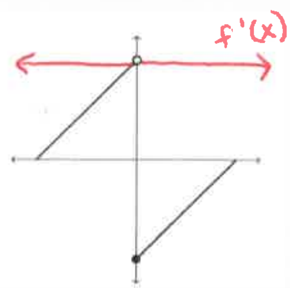
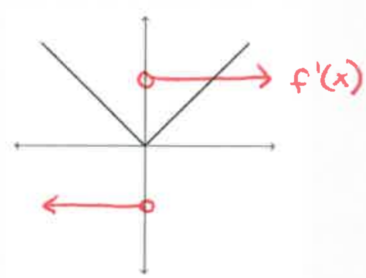
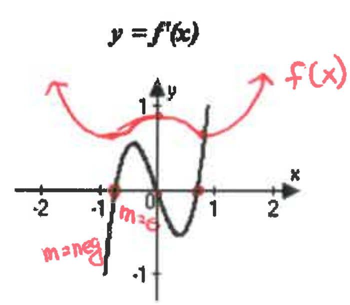
Big Idea #2 Derivatives

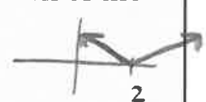
Concept	Question
1. Constructing a Tangent Line	Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$. Write an equation for the line tangent to the curve at the point $(-1, 1)$. $\left.\frac{dy}{dx}\right\|_{(-1,1)} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{4}$ $y - 1 = \frac{1}{4}(x+1)$
2. Constructing a Normal Line	Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$. Write an equation for the normal line to the curve at the point $(-1, 1)$. $\left.\frac{dy}{dx}\right\|_{(-1,1)} = \frac{1}{4}$ $\quad m_{\perp} = -4$ $y - 1 = -4(x+1)$
3. Definitions of Slope $\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$	What is $\lim\limits_{h \to 0} \frac{8(\frac{1}{2} + h)^8 - 8(\frac{1}{2})^8}{h}$ $f(x) = 8x^8$ $f'(x) = 64x^7$ $f'(\frac{1}{2}) = 64(\frac{1}{2})^7 = \frac{1}{2}$
	What is $\lim\limits_{x \to 2} \frac{e^x - e^2}{x - 2}$ $f(x) = e^x$ $f'(x) = e^x$ $f'(2) = e^2$

(or L'hospital's Rule)

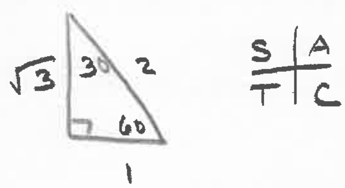
x^{-1}

	If $f(x) = -x^3 + x + \frac{1}{x}$, then $f'(-1) =$ $f'(x) = -3x^2 + 1 - x^{-2}$ $f'(-1) = -3 + 1 - 1 = -3$
4. Basic Derivative	Let f and g be differentiable functions with the following properties I. $g(x) > 0$ for all x II. $f(0) = 1$ If $h(x) = \underbrace{f(x)g(x)}_{\text{would be product rule BUT}}$ and $h'(x) = f(x)g'(x)$, then $f(x) =$ not product rule A. $f'(x)$ B. $g(x)$ C. e^x D. 0 E. 1 so $f(x)$ must be constant $f(0) = 1$ so $f(x) = 1$
5. Graphing the derivative from the original graph	Graph the derivative of the function on the same graph as the function.  
6. Graphing the original function from the derivative graph	Below is the graph of the derivative of f. Graph the original function on the same graph.  * doesn't matter where on y axis

$\frac{(x-2)(x+2)}{\cancel{x-2}} = x+2$

7. One-side Derivatives * if it was cont, to test diff $f'(x) = \{$ $\lim\limits_{x\to 2^-} f'(x) = \lim\limits_{x\to 2^+} f'(x) =$	$f(x) = \begin{cases} \frac{x^2-4}{x-2} & , x \neq 2 \\ 1 & , x = 2 \end{cases}$ Let f be the function defined above. Which of the following statements about f are true? I. f has a limit at $x = 2$ II. f is continuous at $x = 2$ III. f is differentiable at $x = 2$ $\lim\limits_{x\to 2} f(x) = 2+2 = 4$ discontinuous $f(2) = 1$ must be continuous to be diff
8. Determining if a function is differentiable with limits $\lim\limits_{x\to 2^-} f'(x) = c$ $\lim\limits_{x\to 2^+} f'(x) = 4-c$ $c = 4-c$ $c = 2$	$f(x) = \begin{cases} cx+d & , x \leq 2 \\ x^2 - cx & , x > 2 \end{cases}$ $f'(x) = \begin{cases} c, & x \leq 2 \\ 2x-c, & x > 2 \end{cases}$ Let f be the function defined above, where c and d are constants. If f is differentiable at $x = 2$, what is the value of $c + d$? $\lim\limits_{x\to 2^-} f(x) = 2c+d$ $\lim\limits_{x\to 2^+} f(x) = 4-2c$ $2c+d = 4-2c$ $4c+d = 4$ $4(2)+d = 4$ $d = -4$
9. Derivative fails to exist knowing parent graphs *corner, cusp, etc.	Let f be the function given by $f(x) = \lvert x-2 \rvert$. Which of the following statements about f are true?  I. f is continuous at $x = 2$ ✓ II. f is differentiable at $x = 2$ ✗ III. f has no absolute minimum at $x = 2$ ✗ has abs min at 2
10. Differentiability implies continuity	The function f is differentiable at $x = 3$. Which of the following statements is guaranteed to be false? I. $\lim\limits_{x\to 3} f(x)$ exists T II. $\lim\limits_{h\to 0} \frac{f(3+h)-f(3)}{h}$ exists T Def of deriv III. $f(3)$ exists T IV. $f''(3)$ exists V. $\lim\limits_{x\to 3} f(x) = f(3)$ ✓ if diff then cont ↓ $\lim\limits_{x\to 3} f(x) = f(3)$

$\boxed{c+d = -2}$

11. Product Rule	If $f(x) = (x-1)(x^2+3)^3$, then $f'(x) =$ $f'(x) = (x^2+3)^3 + (x-1)(3(x^2+3)^2(2x))$ $= (x^2+3)^3 + 6x(x-1)(x^2+3)^2$ OR $= (x^2+3)^3(6x^2-6x+1)$
12. Quotient Rule	What is the instantaneous rate of change at $x = 2$ of the function f given by $f(x) = \frac{x^2-2}{x-1}$? $f'(x) = \frac{(x-1)(2x) - (x^2-2)}{(x-1)^2}$ $f'(2) = \frac{4-2}{1} = 2$
13. Derivative of all 6 trig functions 	If $f(x) = \tan(2x)$, then $f'(\frac{\pi}{6}) =$ $f'(x) = 2\sec^2 2x$ $f'(\pi/6) = 2\sec^2(\pi/3) = 2(2/\sqrt{3})^2 = 8/3$
	If $f(x) = \sin(e^{-x})$, then $f'(x) =$ $f'(x) = -\cos(e^{-x})$
	If $\overset{f}{\cancel{g}}(x) = \cos(x^2)$, then $f'(x) =$ $f'(x) = -2x\sin(x^2)$
	$\frac{d}{dx}\cot(3x)$ $= -3[\csc(3x)]^2$
	Find the derivative of $y = 3\sec(\pi x)$ $y' = 3\pi \sec \pi x \tan \pi x$

Find the derivative of $f(x) = \csc(5x^2)$

$$f'(x) = -10x\csc(5x^2)\cot(5x^2)$$

14. Chain Rule

$\frac{d}{dx}\cos^2(x^3)$

$$= 2\cos(x^3)\left(-\sin(x^3)\,(3x^2)\right)$$

$$= -6x^2\cos(x^3)\sin(x^3)$$

15. Derivative of Inverse Function

x	$f(x)$	$f'(x)$
1	2	$\frac{1}{2}$
2	3	$\frac{1}{3}$
3	1	-2

$f \rightarrow (3,1)$ so $f^{-1} \rightarrow (1,3)$

If f and f^{-1} exist, are continuous and differentiable for $x > 0$, then $\frac{d}{dx}(f^{-1}(x))$ at $x = 1$ is what?

$$(f^{-1}(1))' = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(3)} = -\frac{1}{2}$$

16. Implicit Differentiation

The slope of the line tangent to the curve $y^2 + (\underbrace{xy}_{\text{product}} + 1)^3 = 0$ at $(2,-1)$ is what?

$$2y\,dy/dx + 3(xy+1)^2\left(y + x\,dy/dx\right) = 0$$

$$2(-1)\,dy/dx + 3(2(-1)+1)^2\left((-1) + 2\,dy/dx\right) = 0$$

$$-2\,dy/dx + 3(-1 + 2\,dy/dx) = 0$$

$$4\,dy/dx = 3$$

$$dy/dx = 3/4$$

17. Derivatives of all 6 inverse trig functions simplify $\frac{-1}{2(1/2)\lvert x\rvert\sqrt{\frac{x^2-4}{4}}}$ cancels $\sqrt{4}=2$ mult by reciprocal $=\frac{-2}{\lvert x\rvert\sqrt{x^2-4}}$	Find the derivative of $\sin^{-1}(x^2)$ $\frac{2x}{\sqrt{1-x^4}}$
	$\frac{d}{dx}\cos^{-1}(3x)$ $-\frac{3}{\sqrt{1-9x^2}}$
	$\frac{d}{dx}\tan^{-1}(e^x)$ $=\frac{e^x}{1+e^{2x}}$
	Find the derivative of $\sec^{-1}(x)$ $=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
	Find the derivative of $\cot^{-1}(5x^2)$ $-\frac{10x}{1+25x^4}$
	$\frac{d}{dx}\csc^{-1}\frac{x}{2}$ $=-\frac{1}{2\lvert x/2\rvert\sqrt{1/4\,x^2-1}}=\frac{-2}{\lvert x\rvert\sqrt{x^2-4}}$
18. Derivative of exponential	If $f(x)=\frac{e^{2x}}{2x}$, then $f'(x)=$ $f'(x)=\frac{2x(2e^{2x})-e^{2x}\cdot 2}{4x^2}$ $=\frac{4xe^{2x}-2e^{2x}}{4x^2}=\frac{2xe^{2x}-e^{2x}}{2x^2}$

OR

$=\frac{e^{2x}(2x-1)}{2x^2}$

	If $f(x) = (x-1)^{\frac{3}{2}} + \frac{e^{x-2}}{2}$, then $f'(2) =$ $f'(x) = \frac{3}{2}(x-1)^{1/2} + \frac{1}{2}e^{x-2}$ $f'(2) = \frac{3}{2} + \frac{1}{2}e^0 = \frac{3}{2} + \frac{1}{2} = \boxed{2}$
19. Derivative of logarithmic	If $f(x) = x^2 + 2x$, then $\frac{d}{dx}(f(\ln x)) =$ $f(\ln x) = (\ln x)^2 + 2\ln x$ $f'(\ln x) = \frac{2\ln x}{x} + \frac{2}{x} = \boxed{\frac{2\ln x + 2}{x}}$
	If $f(x) = (\ln x)^2$, then $f''(\sqrt{e}) =$ $f'(x) = \frac{2\ln x}{x}$ $f''(x) = \frac{x(2/x) - 2\ln x}{x^2}$ $= \frac{2 - 2\ln x}{x^2}$ $f''(\sqrt{e}) = \frac{2 - 2\ln\sqrt{e}}{e} = \frac{2-2(1/2)}{e}$ $\boxed{= \frac{1}{e}}$
20. L'hospital's Rule	Functions f, g, and h are twice differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x-2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$. The function h satisfies $h(x) = \frac{x^2-4}{1-(f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers. $\lim_{x\to 2} 1 - (f(x))^3 = 0$ $1 - (f(2))^3 = 0$ $\boxed{f(2) = 1}$ $h(2) = \lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3}$ L'hospitals $\lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)}$ $h(2) = 4$ so $\lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} = 4$ $\frac{4}{-3(f(2))^2 f'(2)} = 4$

$\frac{4}{f'(2)} = -12$

$\boxed{f'(2) = -\frac{1}{3}}$

21. Derivatives from a table	

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	6	5	3	-2
1	3	-3	-1	2
3	1	-2	2	3

The table above gives values of f, f', g, g' at selected values of x. If $h(x) = f(g(x))$, then $h'(1) =$

$h'(x) = f'(g(x))\, g'(x)$

$h'(1) = f'(g(1))\, g'(1)$

$= f'(-1)(2)$

$= 5(2)$

$= 10$