Big Idea #3 Application of Derivatives

Concept	Question
1. Velocity	A particle moves along the $x-$axis so that its position at time t is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero? $v(t) = x'(t) = 2t - 6$ $0 = 2t - 6$ $\boxed{t = 3}$
2.. Acceleration $\boxed{\text{max acc} = 21 \text{ @ } t = 3}$	What is the maximum acceleration attained on the interval $0 \le t \le 3$ by the particle whose velocity is given by $v(t) = t^3 - 3t^2 + 12t + 4$? $a(t) = 3t^2 - 6t + 12$ $\quad$ $a'(t) = 6t - 6$ $\quad$ $-\ \ +$ $a'(t)$ ★ remember endpoints ★ $\quad$ $0 = 6t - 6$ $a(1) = 9$ $\quad$ $a(3) = 21$ $\quad$ $t = 1$ $\quad$ check actually a min $a(0) = 4$
3. Speed	(Calc Active) A particle moves along a straight line. For $0 \le t \le 5$, the velocity of the particle is given by $v(t) = -2 + (t^2 + 3t)^{6/5} - t^3$, and the position of the particle is given by $s(t)$. It is known that $s(0) = 10$. a. Find all times t in the interval $2 \le t \le 4$ for which the speed of the particle is 2. $\lvert v(t) \rvert = \text{speed}$ $\boxed{t = 3.128,\ 3.473}$ b. Is the speed of the particle increasing or decreasing at time $t = 4$? Give a reason for your answer. $v(4) = -11.476 < 0$ $a(4) = 97.517 > 0$ particle is slowing down at $t = 4$ b/c $v(4) < 0$ and $a(4) > 0$.

4. When a particle changes direction

(Calc Active)
A particle moves along a straight line. For $0 \le t \le 5$, the velocity of the particle is given by $v(t) = -2 + (t^2 + 3t)^{6/5} - t^3$, and the position of the particle is given by $s(t)$. It is known that $s(0) = 10$. Find all times t in the interval $0 \le t \le 5$ at which the particle changes direction. Justify your answer.

$v(t) = 0$

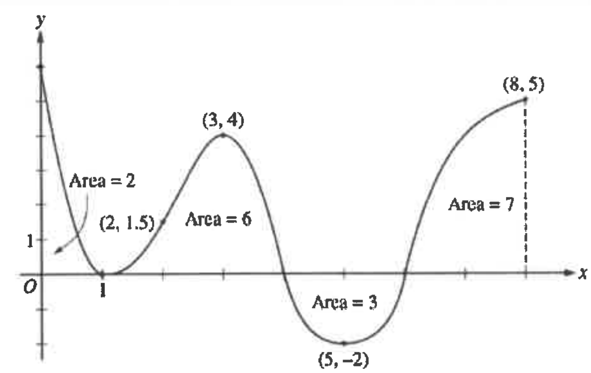
$\boxed{t = 0.536,\ 3.318}$

$-$ $+$ $-$ $v(t)$
0.536   3.318

b/c $v(t)$ changes from neg to pos (left to right) @ $t = 0.536$ and pos to neg (right to left) @ $t = 3.318$

*easier to graph $v(t)$ then plug in test values

5. Absolute Extrema



Graph of f'

+   +   −   + f'
0   1   4   6   8
min

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

a. Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

$\int_0^8 f'(x)\,dx = f(8) - f(0)$

$2 + 6 - 3 + 7 = 4 - f(0)$

$f(0) = 4 - 12$

$= -8$

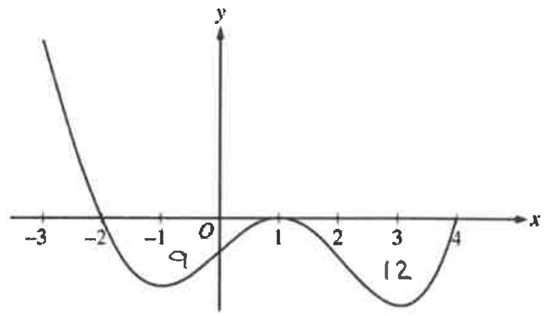
$\int_6^8 f'(x)\,dx = f(8) - f(6)$

$7 = 4 - f(6)$

$f(6) = -3$

$\boxed{f(0) = -8}$

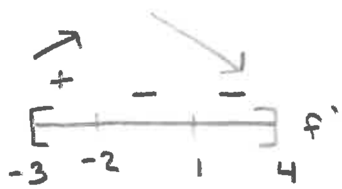
6. Local Extrema



Graph of f'

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

a. Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.



f has a relative max at x = -2

b/c f' changes from pos to neg

at x = -2

7. Mean Value Theorem

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values are given in the table above.

a. Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

$$\frac{f'(1) - f'(-1)}{1 - (-1)} = f''(c)$$

$$\frac{0 - 0}{2} = f''(c)$$

$$0 = f''(c)$$

8. Function Increasing or Decreasing

What are all values of x for which the function f defined by $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing?

$$f'(x) = 3x^2 + 6x - 9$$

$$0 = 3x^2 + 6x - 9$$

$$0 = x^2 + 2x - 3$$

$$0 = (x+3)(x-1)$$

$$x = 1, -3$$

f': $+$ | -3 | $-$ | 1 | $+$

$$\boxed{(-\infty, -3) \cup (1, \infty)}$$

9. First Derivative Test

For what value of x does the function $f(x) = (x-2)(x-3)^2$ have a relative maximum?

$$f'(x) = (1)(x-3)^2 + (x-2)(2(x-3))$$

$$= (x-3)[(x-3) + (x-2)2]$$

$$= (x-3)(3x-7)$$

$$0 = (x-3)(3x-7)$$

$$x = 3,\ 7/3$$

$f'(x)$: $+$ ↗ | $7/3$ | $-$ ↘ | 3 | $+$ ↗

$$\boxed{\text{max @ } x = 7/3}$$

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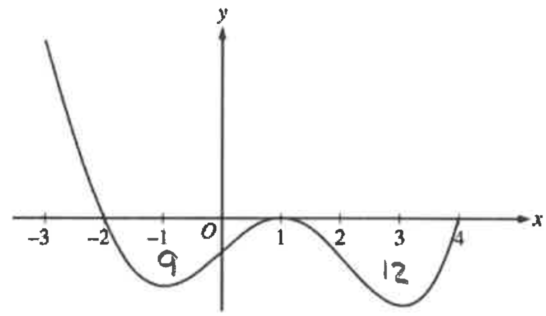

Graph of f'

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3,4]$. The graph of f' has horizontal tangents at $x=-1$, $x=1$, and $x=3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2,1]$ and $[1,4]$ are 9 and 12, respectively.

a. On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.

f'': $-$ | $+$ | $-$ | $+$ at -1, 1, 3

*slope of f'

f': $+$ | $-$ | $-$ at -2, 1, 4

concave down $[-3,-1) \cup (1,3)$

dec $(-2,1) \cup (1,4]$

concave down & dec on $(-2,-1) \cup (1,3)$ b/c $f''<0$ and $f'<0$
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<td>10. Concavity - concave up or down</td>
<td>
The graph of $y = 3x^4 - 16x^3 + 24x^2 + 48$ is concave down for

a. $x < 0$

b. $x > 0$

c. $x < -2$ or $x > -\frac{2}{3}$

d. $x < \frac{2}{3}$ or $x > 2$

e. $\frac{2}{3} < x < 2$

$y' = 12x^3 - 48x^2 + 48x$

$y'' = 36x^2 - 96x + 48$

$0 = 36x^2 - 96x + 48$

$0 = 3x^2 - 8x + 4$

$0 = (3x-2)(x-2)$

$x = 2, \ 2/3$

y'': $+$ | $-$ | $+$ at $2/3$, 2
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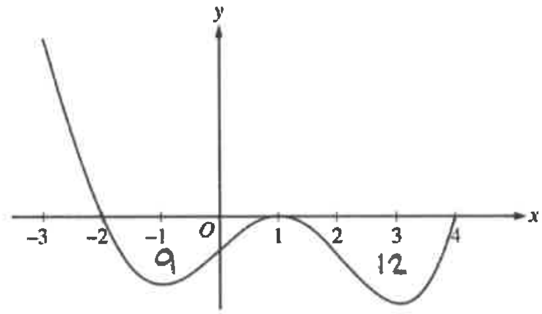
11. Points of Inflection	 Graph of f' The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively. a. Find the x-coordinates for all points of inflection for the graph of f. Give a reason for your answer. $f'' = 0$ and changes sign $x = -1, 1, 3$ candidates $-$ $+$ $-$ $+$ f'' (slope of f') -1 1 3 $x = -1$ and 3 are pts of inflection b/c $f'' = 0$ and f' changes from dec to inc $x = 1$ is a pt of inflection b/c $f'' = 0$ and f' changes from inc to dec
12. Second Derivative Test	What values of x does $f(x) = x^3 - 3x^2 + 12$ have a maximum on the closed interval $[-2, 4]$? $f'(x) = 3x^2 - 6x$ $0 = 3x^2 - 6x$ $0 = 3x(x-2)$ $x = 0, 2$ $\boxed{x = 0}$ $f''(x) = 6x - 6$ $f''(0) = -6$ concave down ↳ max $f''(2) = 6$ concave up ↳ min

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<tr>
<td rowspan="2">13. Optimization
*Open Box
*Area of a field and fencing</td>
<td>A gardener wants to make a rectangular enclosure using a wall as one side and 120 m of fencing for the other three sides. Express the area in terms of x, and find the value of x that gives the greatest area.

$A = xy$
$P = x + 2y$
$120 = x + 2y$
$120 - x = 2y$
$60 - \frac{1}{2}x = y$
$A = x(60 - \frac{1}{2}x)$
$A = 60x - \frac{1}{2}x^2$
$A' = 60 - x$
$0 = 60 - x$
$\boxed{x = 60\text{ m}}$</td>
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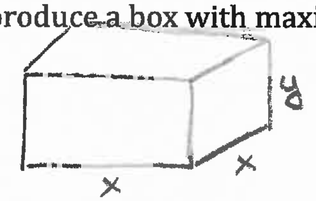
<td>A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

$V = x \cdot x \cdot y$
$= x^2 y$
$S = x^2 + 4xy$
$108 = x^2 + 4xy$
$\frac{108 - x^2}{4x} = y$
$V = x^2\left(\frac{108 - x^2}{4x}\right)$
$= \frac{1}{4}x(108 - x^2)$
$= 27x - \frac{1}{4}x^3$
$V' = 27 - \frac{3}{4}x^2$
$0 = 27 - \frac{3}{4}x^2$
$6 = x$
$\frac{108 - 6^2}{4 \cdot 6} = y$
$3 = y$
$\boxed{6\text{ in by }6\text{ in by }3\text{ in}}$</td>
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<td>14. Linear Approximation
*using a tangent line to approximate a function value</td>
<td>(Calc Active)
Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.
a. For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.
$A(30) = 0.782927$ $A'(30) = -0.055976$
$L(t) - 0.782927 = -0.055976(t - 30)$
$0.5 - 0.782927 = -0.055976(t - 30)$
$\boxed{t = 35.054}$</td>
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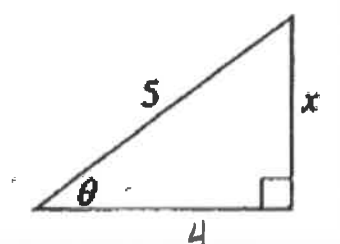
<td>15. Related Rates
*Similar Triangles (lamp post)
*Sliding Ladder
*Cone of water with a leak</td>
<td>

x=3
In the triangle shown above, if θ increases at a constant rate of 3 radians per minutes, at what rate is x increasing in units per minute when $x = 3$ units.
$\frac{d\theta}{dt} = 3$ $\sin\theta = \frac{x}{5}$
$\frac{dx}{dt} = ?$ $\cos\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$
$x = 3$
$\frac{4}{5}(3) = \frac{1}{5}\frac{dx}{dt}$
$12 = dx/dt$</td>
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$\boxed{\frac{dx}{dt} = 12 \text{ units/min}}$

Water is draining from a right conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$)

a. Write an expression for the volume of water in the conical tank as a function of h.

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \left(\tfrac{1}{3}h\right)^2 h$$

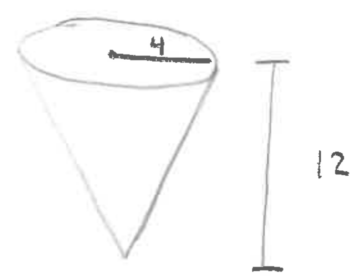
$$= \frac{1}{27}\pi h^3$$

b. At what rate is the volume of water in the conical tank changing when $h = 3$?

$$\frac{dV}{dt} = \frac{3}{27}\pi h^2 \frac{dh}{dt}$$

$$= \frac{1}{9}\pi (3)^2(-9)$$

$$= -9\pi \ ft^3/min$$



$$\frac{dh}{dt} = h - 12$$

$$\left.\frac{dh}{dt}\right|_{h=3} = -9$$

$$\frac{r}{h} = \frac{4}{12}$$

$$r = \frac{1}{3}h$$