The derivative of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

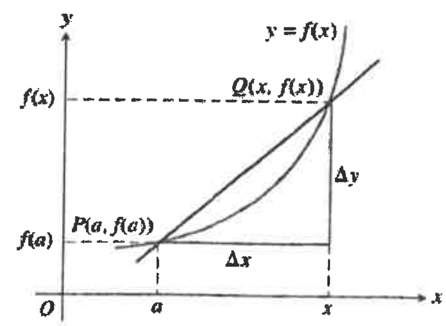
provided the limit exists

Alternative Derivative (derivative at a point)

The derivative of the function f at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

provided the limit exists.



1. $\lim_{h \to 0} \frac{\cos(x+h)-\cos x}{h}$

$f(x) = \cos x$

$\boxed{f'(x) = -\sin x}$

2. $\lim_{h \to 0} \frac{5(x+h)^4-5x^4}{h}$

$f(x) = 5x^4$

$\boxed{f'(x) = 20x^3}$

3. $\lim_{h \to 0} \frac{10(2+h)^3-80}{h}$

$f(x) = 10x^3$

$f'(x) = 30x^2$

$f'(2) = 30(2)^2 = \boxed{120}$

4. $\lim_{h \to 0} \frac{(3+h)^4-81}{h}$

$f(x) = x^4$

$f'(x) = 4x^3$

$f'(3) = 4(3)^3 = \boxed{108}$

5. $\lim_{h \to 0} \frac{\sec(x+h)-\sec x}{h}$

$f(x) = \sec x$

$\boxed{f'(x) = \sec x \tan x}$

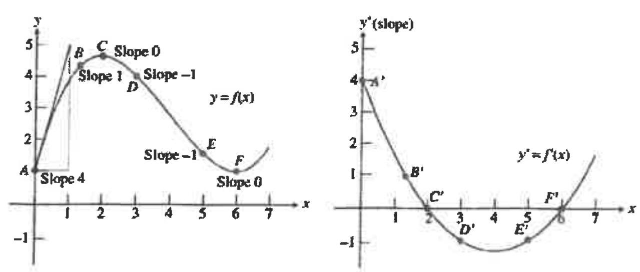
6. $\lim_{h \to 0} \frac{\sin(\frac{\pi}{6}+h)-\frac{1}{2}}{h}$

$f(x) = \sin x$

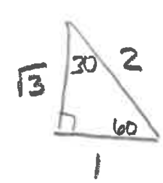
$f'(x) = \cos x$

$f'(\tfrac{1}{2}) = \cos \pi/6$

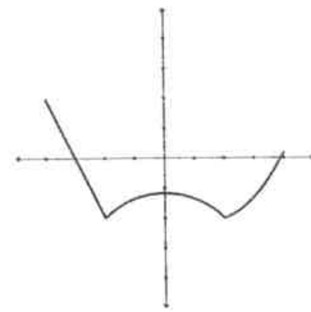
$\boxed{= \frac{\sqrt{3}}{2}}$



(a) (b)



1. Consider the graph of $f(x)$ below to answer the following questions:



   a. Are there any x values for which the derivative does not exist?

   $x = -2, 2$

   b. Are there any x values for which $f'(x) = 0$?

   $x = 0$

   c. This particular function f has an interval on which its derivative $f'(x)$ is constant. What is this interval? What does the derivative function look like there? Estimate the slope of $f(x)$ on that interval.

   $(-4, -2)$

   horizontal line at $y = -2$

   $m = \frac{2-(-2)}{-4-(-2)} = \frac{4}{-2} = -2$

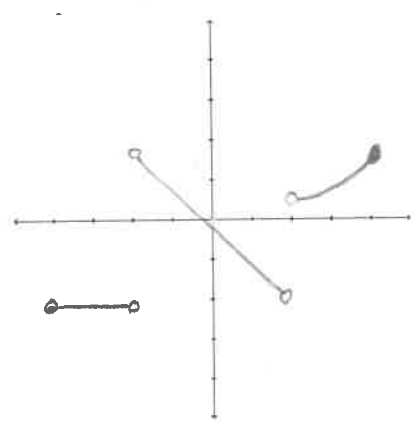
   d. On which interval or intervals is $f'(x)$ positive?

   $(-2, 0) \cup (2, 4)$

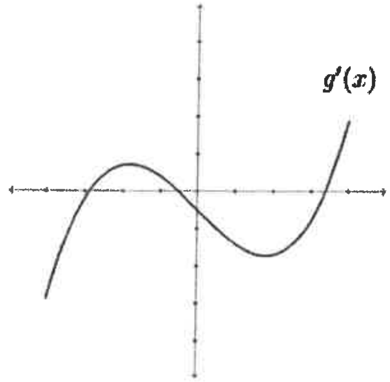
   e. On which interval or intervals is $f'(x)$ negative?

   $(-4, -2) \cup (0, 2)$

   f. Sketch a graph of the derivative of the function.



2. Below is a graph of a derivative $g'(x)$. Assume this is the entire graph of $g'(x)$. Use the graph to answer the following questions about the original function $g(x)$.



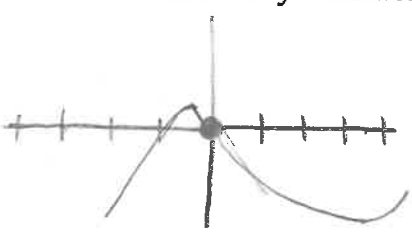
a. On which interval(s) is the original function $g(x)$ increasing?

$(-3, -\frac{1}{2}) \cup (\frac{7}{2}, 4)$

b. On which interval(s) is the original function $g(x)$ decreasing?

$(-4, -3) \cup (-\frac{1}{2}, \frac{7}{2})$

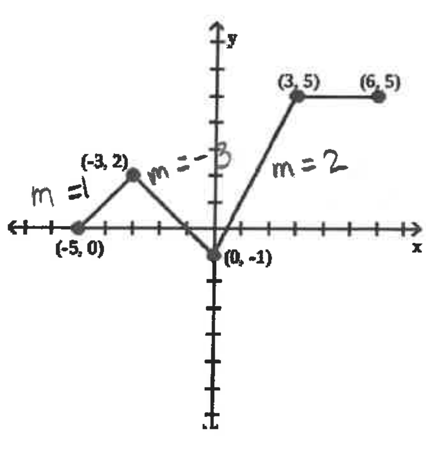
c. Now suppose that $g(0) = 0$. Is the function $g(x)$ ever positive? That is, is there any x values such that $g(x) \geq 0$? How do you know?

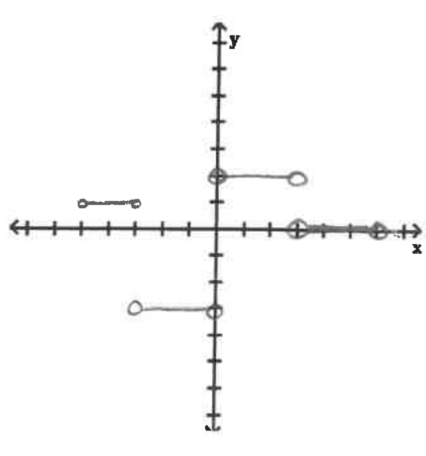


yes b/c from $(-\frac{1}{2}, 0)$ the function $g(x)$ is decreasing so it must be positive to have the point $(0, 0)$

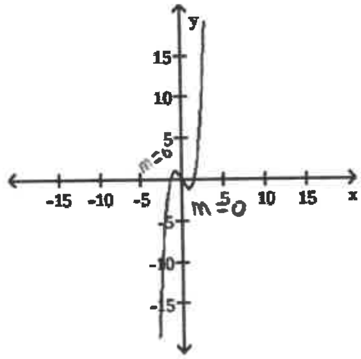
3. Graph the derivative of the function below:

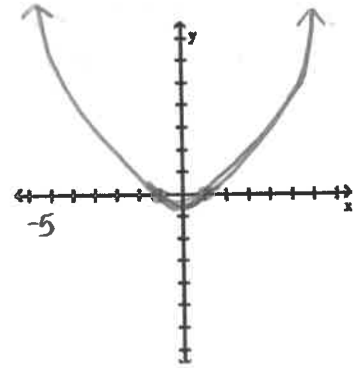
a.



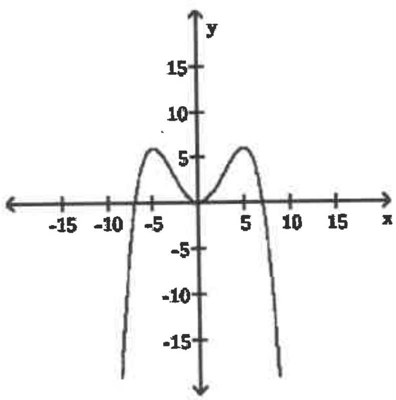


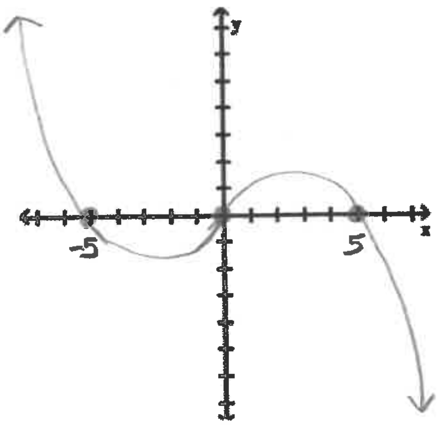
b.





c.





One-Sided Derivatives

A function $y = f(x)$ is **differentiable on a closed interval** $[a, b]$ if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \quad \text{[the right-hand derivative at } a\text{]}$$

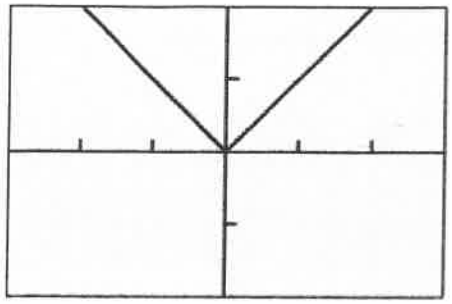
$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \quad \text{[the left-hand derivative at } b\text{]}$$

exist at the endpoints. In the right-hand derivative, h is positive and $a + h$ approaches a from

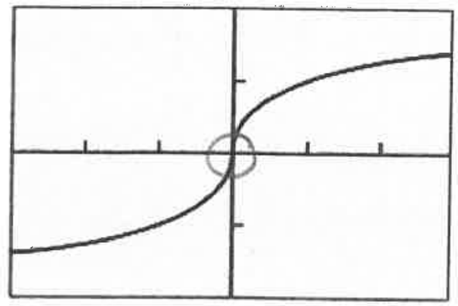
Where $f'(a)$ Does Not Exist

A function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines, $\dfrac{f(x)-f(a)}{x-a}$ fail to approach a limit as x approaches a.

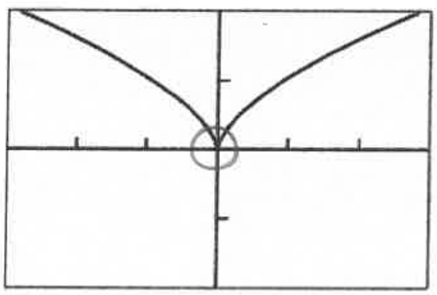
1. CORNER: absolute value



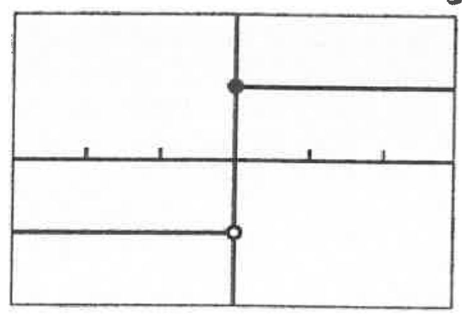
3. VERTICAL TANGENT $\sqrt[3]{x}$



2. CUSP: $x^{2/3}$



4. DISCONTINUITY jump



1. For each function, $f(x)$, determine whether the function is continuous and/or differentiable.

a. $$f(x) = \begin{cases} x^2, & x \geq 0 \\ x, & x < 0 \end{cases}$$

continuous ✓

$\lim_{x \to 0^-} f(x) = 0$ $\lim_{x \to 0^+} f(x) = 0$

$f(0) = 0$

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ 1 & x < 0 \end{cases}$$

Diff

$\lim_{x \to 0^-} f'(x) = 1$ $\lim_{x \to 0^+} f'(x) = 0$

Continuous & Not Diff.

b. $$f(x) = \begin{cases} 4 - x^2, & x < 1 \\ 2x + 2, & x \geq 1 \end{cases}$$

cont

$\lim_{x \to 1^-} f(x) = -2$

$\lim_{x \to 1^+} f(x) = 2$

$$f'(x) = \begin{cases} -2x & x < 1 \\ 2 & x \geq 1 \end{cases}$$

not continuous so not diff.

c. $$f(x)=\begin{cases}\sqrt{x}-3, x>1\\ \frac{1}{2}x-\frac{5}{2}, x\le 1\end{cases}$$

cont

$\lim_{x\to 1^-} f(x) = -2$ $\lim_{x\to 1^+} f(x) = -2$

$$f'(x) = \begin{cases}\frac{1}{2}x^{-1/2}\\ \frac{1}{2}\end{cases}$$

diff

$\lim_{x\to 1^-} f'(x) = \frac{1}{2}$ $\lim_{x\to 1^+} f'(x) = \frac{1}{2}$

$\boxed{\text{Continuous and Diff.}}$

2. Find the values of a and b that make the function below differentiable.

$$f(x)=\begin{cases}ax^2+10, & x\ge 2\\ x^2-6x+b, x<2\end{cases}$$

$\lim_{x\to 2^-} f(x) = 2^2-6(2)+b$
$= -8+b$

$\lim_{x\to 2^+} f(x) = 4a+10$

$-8+b = 4a+10$

$$f'(x) = \begin{cases}2ax & x\ge 2\\ 2x-6 & x<2\end{cases}$$

$\lim_{x\to 2^-} f'(x) = 2(2)-6$
$= -2$

$\lim_{x\to 2^+} f'(x) = 4a$

$-2 = 4a$
$-\frac{1}{2} = a$

$-8+b = 4(-\frac{1}{2})+10$
$b = 16$

$\boxed{a = -\frac{1}{2},\ b = 16}$

THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = a$, then f is continuous at $x = a$.

*****Not an If And Only If Statement*******

think of $|x|$

THEOREM 2 Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

1.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	–1	1	2
2	1	$\frac{1}{2}$	3	0
3	3	2	1	–2

Given $h(x) = f(x) \cdot g(x)$, find $h'(3)$

$h'(x) = f'(x)g(x) + g'(x)f(x)$

$h'(3) = f'(3)g(3) + g'(3)f(3)$

$= 2(1) + (-2)(3)$

$= 2 - 6$

$\boxed{= -4}$

2.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	–1	2	–1
2	2	–1	1	0
3	1	–1	2	1

Given $h(x) = \frac{f(x)}{g(x)}$, find $h'(3)$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

$$h'(3) = \frac{2(-1) - (1)(1)}{2^2}$$

$$= \frac{-2-1}{4} = \boxed{-\frac{3}{4}}$$

DEFINITION Instantaneous Velocity

The (instantaneous) velocity is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

DEFINITION Acceleration

Acceleration is the derivative of velocity with respect to time. If a body's velocity at time t is $v(t) = ds/dt$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

CALCULUS AB
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. For $0 \le t \le 6$, a particle is moving along the x-axis. The particle's position, $x(t)$, is not explicitly given. The velocity of the particle is given by $v(t) = 2\sin\left(e^{t/4}\right) + 1$. The acceleration of the particle is given by $a(t) = \frac{1}{2}e^{t/4}\cos\left(e^{t/4}\right)$ and $x(0) = 2$.

(a) Is the speed of the particle increasing or decreasing at time $t = 5.5$? Give a reason for your answer.

(b) Find the average velocity of the particle for the time period $0 \le t \le 6$.

(c) Find the total distance traveled by the particle from time $t = 0$ to $t = 6$.

(d) For $0 \le t \le 6$, the particle changes direction exactly once. Find the position of the particle at that time.

a) $v(5.5) = -0.453$

$a(5.5) = -1.3585$

The particle is increasing b/c both velocity and acceleration are negative at $t = 5.5$

b) $\int_0^6 \frac{1}{6} v(t)\,dt = 1.949$

c) $\int_0^6 |v(t)|\,dt = 12.573$

d) $0 = v(t)$

$t = 5.196$

$+ \quad\quad - \quad v(t)$

5.196

v(t) changes from pos to neg at $t = 5.196$

$x(5.196) = x(0) + \int_0^{5.196} v(t)\,dt$

$x(5.196) = 2 + \int_0^{5.196} v(t)\,dt$

$= 14.135$