RULE 8 The Chain Rule

If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

1. Find the derivative of the following:
   a. $y = 2x^4\sqrt{x^2 - 5}$

$$y' = 8x^3\sqrt{x^2-5} + 2x^4\left(\tfrac{1}{2}(x^2-5)^{-1/2}(2x)\right)$$

$$= 8x^3\sqrt{x^2-5} + \frac{2x^5}{\sqrt{x^2-5}}$$

BC Topic!

Slopes of Parametrized Curves

A parametrized curve $(x(t), y(t))$ is *differentiable at t* if x and y are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Finding dy/dx Parametrically

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (3)$$

Examples:

1. Find the line tangent to the right-hand hyperbola branch defined parametrically by $x = \sec t,\ y = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$ at the point $(\sqrt{2},\ 1)$ where $t = \frac{\pi}{4}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $\qquad \frac{dy}{dt} = \sec^2 t$ $\qquad \frac{dx}{dt} = \sec t \tan t$

$\left.\frac{dy}{dt}\right|_{t=\pi/4} = \sec^2 \pi/4 = (\sqrt{2})^2 = 2$

$\left.\frac{dx}{dt}\right|_{t=\pi/4} = \sec \pi/4 \tan \pi/4 = \sqrt{2}$

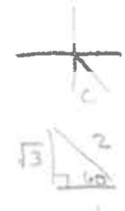
$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\boxed{y - 1 = \sqrt{2}(x - \sqrt{2})}$

2. Find the equation of the line tangent to the curve defined by $x = \sin 2\pi t,\ y = \cos 2\pi t$ at the point $t = -\frac{1}{6}$.

$\frac{dy}{dx} = \frac{-2\pi \sin 2\pi t}{2\pi \cos 2\pi t} = -\tan 2\pi t$

$\left.\frac{dy}{dx}\right|_{t=-1/6} = -\tan -2\pi/6 = -\tan -\pi/3 = \sqrt{3}$



$x = \sin -\pi/3 = -\sqrt{3}/2$

$y = \cos -\pi/3 = \frac{1}{2}$

$(-\sqrt{3}/2,\ 1/2)$

$\boxed{y - 1/2 = \sqrt{3}(x + \sqrt{3}/2)}$

1. Find $\frac{dy}{dx}$ of $x^2 - xy + 3y^2 = 7$

$2x - \left(y + x\, dy/dx\right) + 6y \frac{dy}{dx} = 0$

$2x - y - x\, dy/dx + 6y\, dy/dx = 0$

$\frac{dy}{dx}(-x + 6y) = -2x + y$

$\boxed{\frac{dy}{dx} = \frac{-2x + y}{-x + 6y}}$

2. Given $x^2 + y^2 = 16$, find the equation of the tangent line at $(3,\ \sqrt{7})$.

$2x + 2y\, dy/dx = 0$

$2(3) + 2\sqrt{7}\, dy/dx = 0$

$\frac{dy}{dx} = -\frac{3}{\sqrt{7}} = -\frac{3\sqrt{7}}{7}$

$\boxed{y - \sqrt{7} = -\frac{3\sqrt{7}}{7}(x - 3)}$

Challenge Problems!

1. Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$

$$6x^2 - 6y\, dy/dx = 0$$

$$\frac{dy}{dx} = \frac{x^2}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y(2x) - x^2\, dy/dx}{y^2}$$

$$= \frac{2xy - x^2(x^2/y)}{y^2}$$

$$= \frac{2xy - x^4/y}{y^2}$$

$$= \frac{\frac{2xy^2 - x^4}{y}}{y^2}$$

$$= \frac{2xy^2 - x^4}{y} \cdot \frac{1}{y^2} = \boxed{\frac{2xy^2 - x^4}{y^3}}$$

Theorem 1 **Derivatives of Inverse Functions**

If f is differentiable at every point of an interval I and $f'(x)$ is never zero on I, then f has an inverse, and f^{-1} is differentiable at every point of the interval $f(I)$. If $f^{-1}(a) = b$, the **inverse function slope relationship** relates the derivative by the equation

$$f^{-1\prime}(a) = \frac{1}{f'(b)}$$

1.

Let f and g be inverse functions.

The following table lists a few values of f, g, and f'.

x	$f(x)$	$g(x)$	$f'(x)$
-2	2	5	1
5	-2	-4	$-\frac{1}{2}$

$(-2, 5)$

$g'(-2) = \boxed{-2}$

$g'(-2) = \frac{1}{f'(5)}$

$= \frac{1}{-1/2}$

$= -2$

2.

Let $h(x) = 7 - x - 2x^5$ and let f be the inverse function of h. Notice that $h(-1) = 10$.

$f'(10) = \boxed{-1/11}$

$f'(10) = \frac{1}{h'(-1)}$

$h'(x) = -1 - 10x^4$

$h'(-1) = -1 - 10(-1)^4$

$= -11$

$= \frac{1}{-11}$

3.

Let $g(x) = x^5 + 3x$ and let h be the inverse function of g. Notice that $g(1) = 4$.

$h'(4) = \boxed{1/8}$

$h'(4) = \frac{1}{g'(1)}$

$g'(x) = 5x^4 + 3$

$g'(1) = 8$

$= 1/8$

4.

Let f and g be inverse functions.

The following table lists a few values of f, g, and g'.

x	$f(x)$	$g(x)$	$g'(x)$
-3	5	4	$-\frac{1}{4}$
4	-3	2	2

$(4, -3)$

$f'(4) = \boxed{-4}$

$f'(4) = \frac{1}{g'(-3)}$

$= \frac{1}{-1/4}$

$= -4$

Derivative Rules:

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\cdot\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1}x)=-\frac{1}{|x|\cdot\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$$

$$\frac{d}{dx}(e^x)=e^x$$

$$\frac{d}{dx}(a^x)=a^x\cdot\ln a$$

$$\frac{d}{dx}(\ln x)=\frac{1}{x},\qquad x>0$$

$$\frac{d}{dx}(\ln|x|)=\frac{1}{x},\qquad x\neq 0$$

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\cdot\ln a},\qquad x>0$$

Examples:

1. Find the derivative of y with respect to the appropriate variable.
   a. $y=\sin^{-1}\sqrt{2t}$

$$y'=\frac{1}{\sqrt{1-(\sqrt{2t})^2}}\left(\tfrac{1}{2}(2t)^{-1/2}(2)\right)$$

$$=\frac{(2t)^{-1/2}}{\sqrt{1-2t}}$$

   b. $y=x\sin^{-1}x+\sqrt{1-x^2}$

$$y'=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}+\frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}$$

$$=\sin^{-1}x$$

1. Find the derivative of the following equations:

a. $y = \ln(x^3 + 3x - 2)$

$$y' = \frac{1}{x^3+3x-2}(3x^2+3)$$

$$= \frac{3x^2+3}{x^3+3x-2}$$

b. $y = e^{3x^2}$

$$y' = 6x\, e^{3x^2}$$

c. $y = 3^{x+2}$

$$y' = \ln 3\,(3^{x+2})$$

$$= 3^{x+2}\ln 3$$

d. $y = x^{\sin x}$

$$\ln y = \sin x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x}$$

$$\frac{dy}{dx} = y\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

$$= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$$