SOLUTIONS TO PRACTICE PROBLEM SET 17

1. $\frac{3}{4}$

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x)=\sin 3x$, and $g(x)=\sin 4x$, and $\sin 0=0$. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 4x}=\lim\limits_{x\to 0}\frac{3\cos 3x}{4\cos 4x}$. If we take the new limit, we get $\lim\limits_{x\to 0}\frac{3\cos 3x}{4\cos 4x}=\frac{3\cos 0}{4\cos 0}=\frac{(3)(1)}{(4)(1)}=\frac{3}{4}$.

2. -1

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x)=x-\pi$ and $g(x)=\sin x$. We can see that $x-\pi=0$ when $x=\pi$, and that $\sin\pi=0$. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim\limits_{x\to\pi}\frac{x-\pi}{\sin x}=\lim\limits_{x\to\pi}\frac{1}{\cos x}$. If we take the new limit, we get $\lim\limits_{x\to\pi}\frac{1}{\cos x}=\frac{1}{\cos\pi}=\frac{1}{-1}=-1$.

3. $\frac{1}{6}$

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x)=x-\sin x$ and $g(x)=x^3$. We can see that $x-\sin x=0$ when $x=0$, and that $0^3=0$. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}=\lim\limits_{x\to 0}\frac{1-\cos x}{3x^2}$.

If we take the new limit, we get $\lim\limits_{x\to 0}\frac{1-\cos x}{3x^2}=\frac{1-\cos 0}{3(0)^2}=\frac{1-1}{0}=\frac{0}{0}$. But this is still indeterminate, so what do we do? Use L'Hôpital's Rule again! We take the derivative of the numerator and the denominator: $\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}$. If we take the limit, we get $\lim_{x\to 0}\frac{\sin x}{6x}=\frac{\sin 0}{6(0)}=\frac{0}{0}$. We need to use L'Hôpital's Rule one more time. We get $\lim_{x\to 0}\frac{\sin x}{6x}=\lim_{x\to 0}\frac{\cos x}{6}$. Now, if we take the limit, we get $\lim_{x\to 0}\frac{\cos x}{6}=\frac{\cos 0}{6}=\frac{1}{6}$. Notice that if L'Hôpital's Rule results in an indeterminate form, we can use the rule again and again (but not infinitely often).

4. -2

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x)=e^{3x}-e^{5x}$ and $g(x)=x$. We can see that $e^{3x}-e^{5x}=0$ when $x=0$, and that the denominator is obviously zero at $x=0$. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim_{x\to 0}\frac{e^{3x}-e^{5x}}{x}=\lim_{x\to 0}\frac{3e^{3x}-5e^{5x}}{1}$. If we take the new limit, we get $\lim_{x\to 0}\frac{3e^{3x}-5e^{5x}}{1}=\frac{3e^0-5e^0}{1}=\frac{3(1)-5(1)}{1}=-2$.

5. -2

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x)=\tan x-x$ and $g(x)=\sin x-x$. We can see that $\tan x-x=0$ and $\sin x-x=0$ when $x=0$. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim_{x\to 0}\frac{\tan x-x}{\sin x-x}=\lim_{x\to 0}\frac{\sec^2 x-1}{\cos x-1}$. If we take the new limit, we get $\lim_{x\to 0}\frac{\sec^2 x-1}{\cos x-1}=\frac{\sec^2(0)-1}{\cos(0)-1}=\frac{1-1}{1-1}=\frac{0}{0}$.

But this is still indeterminate, so what do we do? Use L'Hôpital's Rule again! We take the derivative of the numerator and the denominator: $\lim\limits_{x\to 0}\dfrac{\sec^2 x-1}{\cos x-1}=\lim\limits_{x\to 0}\dfrac{2\sec x(\sec x\tan x)}{-\sin x}=\lim\limits_{x\to 0}\dfrac{2\sec^2 x\tan x}{-\sin x}$.

If we take the limit, we get $\lim\limits_{x\to 0}\dfrac{2\sec^2(0)\tan(0)}{-\sin(0)}=\dfrac{0}{0}$. At this point you might be getting nervous as the derivatives start to get messier. Let's try using trigonometric identities to simplify the limit.

If we rewrite the numerator in terms of sin x and cos x, we get $\dfrac{2\sec^2 x\tan x}{-\sin x}=\dfrac{2\left(\dfrac{1}{\cos^2 x}\right)\left(\dfrac{\sin x}{\cos x}\right)}{-\sin x}$.

We can simplify this to $\dfrac{2\left(\dfrac{1}{\cos^2 x}\right)\left(\dfrac{\sin x}{\cos x}\right)}{-\sin x}=-2\dfrac{\dfrac{\sin x}{\cos^3 x}}{\sin x}$. This simplifies to $-2\dfrac{\sin x}{\cos^3 x}\dfrac{1}{\sin x}=\dfrac{-2}{\cos^3 x}$.

Now, if we take the limit, we get $\lim\limits_{x\to 0}\dfrac{-2}{\cos^3 x}=\dfrac{-2}{\cos^3(0)}=\dfrac{-2}{1^3}=-2$.

Note that we could have used trigonometric identities on either of the first two limits as well. Remember that when you have a limit that is an indeterminate form, you can sometimes use algebra or trigonometric identities (or both) to simplify the limit. Sometimes this will get rid of the problem.

6. 0

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$. Here $f(x)=x^5$ and $g(x)=e^{5x}$. We can see that $x^5=\infty$ when $x=\infty$, and that $e^\infty=\infty$. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim\limits_{x\to\infty}\dfrac{x^5}{e^{5x}}=\lim\limits_{x\to\infty}\dfrac{5x^4}{5e^{5x}}=\lim\limits_{x\to\infty}\dfrac{x^4}{e^{5x}}$. If we take the new limit, we get $\lim\limits_{x\to 0}\dfrac{x^4}{e^{5x}}=\dfrac{\infty}{\infty}$. But this is still indeterminate, so what do we do? Use L'Hôpital's Rule again! We take the derivative of the numerator and the denominator: $\lim_{x\to\infty}\frac{x^4}{e^{5x}}=\lim_{x\to\infty}\frac{4x^3}{5e^{5x}}$. We are going to need to use L'Hôpital's Rule again. In fact, we can see that each time we use the rule we are reducing the power of the x in the numerator and that we are going to need to keep doing so until the x term is gone. Let's take the derivative: $\lim_{x\to\infty}\frac{4x^3}{5e^{5x}}=\lim_{x\to\infty}\frac{12x^2}{25e^{5x}}$. And again: $\lim_{x\to\infty}\frac{12x^2}{25e^{5x}}=\lim_{x\to\infty}\frac{24x}{125e^{5x}}$. And one more time: $\lim_{x\to\infty}\frac{24x}{125e^{5x}}=\lim_{x\to\infty}\frac{24}{625e^{5x}}$. Now, if we take the limit, we get $\lim_{x\to\infty}\frac{24}{625e^{5x}}=0$. Notice that if L'Hôpital's Rule results in an indeterminate form, we can use the rule again and again (but not infinitely often).

7. $\frac{1}{7}$

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x)=x^5+4x^3-8$ and $g(x)=7x^5-3x^2-1$. We can see that $x^5+4x^3-8=\infty$ and $7x^5-3x^2-1=\infty$ when $x=\infty$. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim_{x\to\infty}\frac{x^5+4x^3-8}{7x^5-3x^2-1}=\lim_{x\to\infty}\frac{5x^4+12x^2}{35x^4-6x}$. If we take the new limit, we get $\lim_{x\to\infty}\frac{5x^4+12x^2}{35x^4-6x}=\frac{\infty}{\infty}$. But this is still indeterminate, so what do we do? Use L'Hôpital's Rule again! We take the derivative of the numerator and the denominator: $\lim_{x\to\infty}\frac{5x^4+12x^2}{35x^4-6x}=\lim_{x\to\infty}\frac{20x^3+24x}{140x^3-6}$. We are going to need to use L'Hôpital's Rule again. In fact, we can see that each time we use the rule we are reducing the power of the x terms in the numerator and denominator and that we are going to need to keep doing so until the x terms are gone. Let's take the derivative:

$\lim_{x\to\infty}\frac{20x^3+24x}{140x^3-6}=\lim_{x\to\infty}\frac{60x^2+24}{420x^2}$. And again, $\lim_{x\to\infty}\frac{60x^2+24}{420x^2}=\lim_{x\to\infty}\frac{120x}{840x}$. Now we can take the limit: $\lim_{x\to\infty}\frac{120x}{840x}=\lim_{x\to\infty}\frac{1}{7}=\frac{1}{7}$. Notice that if L'Hôpital's Rule results in an indeterminate form, we can use the rule again and again (but not infinitely often).

8. 1

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x)=\ln(\sin x)$ and $g(x)=\ln(\tan x)$, and both of these approach infinity as x approaches 0 from the right. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim_{x\to 0^+}\frac{\ln(\sin x)}{\ln(\tan x)}=\lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}}$. Let's use trigonometric identities to simplify the limit: $\frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}}=\frac{\cos x}{\sin x}\frac{\tan x}{\sec^2 x}=\frac{\cos x}{\sin x}\frac{\frac{\sin x}{\cos x}}{\sec^2 x}=\frac{1}{\sec^2 x}$. Now, if we take the new limit, we get $\lim_{x\to 0^+}\frac{1}{\sec^2 x}=1$.

9. $\frac{1}{2}$

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim\limits_{x\to c}\frac{f(x)}{g(x)}=\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$. Here $f(x) = \cot 2x$ and $g(x) = \cot x$, and both of these approach infinity as x approaches 0 from the right. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim\limits_{x\to 0^+}\frac{\cot 2x}{\cot x}=\lim\limits_{x\to 0^+}\frac{-2\cot 2x\csc 2x}{-\cot x\csc x}$. This seems to be worse than what we started with. Instead, let's use trigonometric identities to simplify the limit: $\frac{\cot 2x}{\cot x}=\frac{\frac{\cos 2x}{\sin 2x}}{\frac{\cos x}{\sin x}}=\frac{\cos 2x}{\sin 2x}\frac{\sin x}{\cos x}=\frac{\cos^2 x-\sin^2 x}{2\sin x\cos x}\frac{\sin x}{\cos x}$. Notice that our problem is with $\sin x$ as x approaches zero (because it becomes zero), not with $\cos x$. As long as we are multiplying the numerator and denominator by $\sin x$, we are going to get an indeterminate form. So, thanks to trigonometric identities, we can eliminate the problem term: $\frac{\cos^2 x-\sin^2 x}{2\sin x\cos x}\frac{\sin x}{\cos x}=\frac{\cos^2 x-\sin^2 x}{2\cos^2 x}$. If we take the limit of this expression, it is not indeterminate. We get $\lim\limits_{x\to 0^+}\frac{\cos^2 x-\sin^2 x}{2\cos^2 x}=\frac{\cos^2 0-\sin^2 0}{2\cos^2 0}=\frac{1^2-0}{2(1^2)}=\frac{1}{2}$. Notice that we didn't need to use L'Hôpital's Rule here. You should bear in mind that just because a limit is indeterminate does not mean that the best way to evaluate it is with L'Hôpital's Rule.

10. 1

Recall L'Hôpital's Rule: If $f(c)=g(c)=0$, or if $f(c)=g(c)=\infty$, and if $f'(c)$ and $g'(c)$ exist, and if $g'(c)\neq 0$, then $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$. Here $f(x)=x$ and $g(x)=\ln(x+1)$, and both of these approach zero as x approaches 0 from the right. This means that we can use L'Hôpital's Rule to find the limit. We take the derivative of the numerator and the denominator: $\lim\limits_{x\to 0^+}\dfrac{1}{\frac{1}{x+1}}=\lim\limits_{x\to 0^+}(x+1)$. Now, if we take the new limit, we get $\lim\limits_{x\to 0^+}(x+1)=1$.

SOLUTIONS TO UNIT 2 DRILL

1. $y-4=-(x-2)$

Remember that the equation of a line through a point (x_1, y_1) with slope m is $y-y_1=m(x-x_1)$. We find the y-coordinate by plugging $x = 2$ into the equation $y=\sqrt{8x}$, and we find the slope by plugging $x = 2$ into the derivative of the equation.

First, we find the y-coordinate, $y_1 : y=\sqrt{8(2)}=4$. This means that the line passes through the point $(2, 4)$.

Next, we take the derivative: $\dfrac{dy}{dx}=\dfrac{4}{\sqrt{8x}}$. Now, we can find the slope, m: $\left.\dfrac{dy}{dx}\right|_{x=2}=\dfrac{4}{\sqrt{8(2)}}=1$. However, this is the slope of the *tangent* line. The *normal* line is perpendicular to the tangent line, so its slope will be the negative reciprocal of the tangent line's slope. In this case, the slope of the normal line is $\dfrac{-1}{1}=-1$. Finally, we plug in the point $(2, 4)$ and the slope $m=-1$ to get the equation of the normal line: $y-4=-(x-2)$.