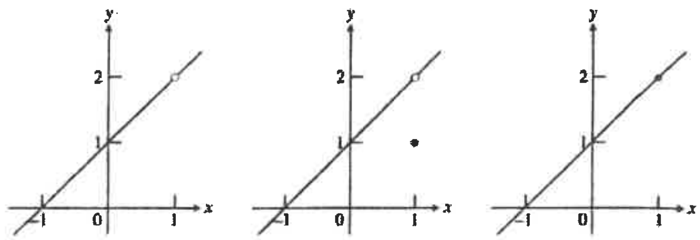
Limit → the value of f(x) as x approaches a certain #

* $f(a) \neq \lim_{x \to a} f(x)$

could be equal but not necessary

$$\lim_{x \to c} f(x) = L$$



(a) $f(x) = \frac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x)$

THEOREM 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$

   The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

   $$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number.

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

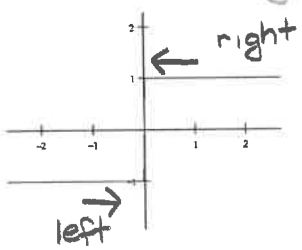
Facts:

★ $\lim_{x\to 0}\dfrac{\sin x}{x}=1$ ★ $\qquad \lim_{x\to 0}\dfrac{1-\cos x}{x}=0 \qquad \lim_{x\to 0}\dfrac{\tan x}{x}=1$

$\lim_{x\to 0}\dfrac{\sin ax}{bx}=\dfrac{a}{b} \qquad \lim_{x\to 0}\dfrac{1-\cos ax}{bx}=0 \qquad \lim_{x\to 0}\dfrac{\tan ax}{bx}=\dfrac{a}{b}$

Limits that Do Not Exist

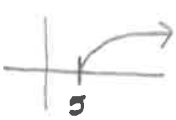
The left and right limits are different.



$\lim_{x\to 0}\dfrac{|x|}{x}=$ DNE

Right Hand: $\lim_{x\to c^+} f(x)$

Left Hand: $\lim_{x\to c^-} f(x)$



1. $\lim_{x\to 5^+}\sqrt{x-5}=0$

2. $\lim_{x\to 5^-}\sqrt{x-5}=$ DNE

3. $\lim_{x\to 5}\sqrt{x-5}=$ DNE

4. $\lim_{x\to 2}\sqrt{16-x^2}=\sqrt{16-2^2}$
$=\sqrt{12}=2\sqrt{3}$

5. $\lim_{x\to 5}\sqrt{16-x^2}=\sqrt{16-5^2}$ $\to \sqrt{-9}$ DNE
★ 1/2 circle ★

6. $\lim_{x\to 4}\sqrt{16-x^2}=\sqrt{16-4^2}$
$\sqrt{(4-x)(4+x)}$ $=\sqrt{0}$ "works" but need to know graph
DNE

7. Use the piecewise function $f(x)=\begin{cases}3-x & x<-3\\ 2x+12 & -3\le x<4\\ 9 & x\ge 4\end{cases}$ for the limits below:

a. $\lim_{x\to -3^+} f(x)=2(-3)+12$
$=6$

b. $\lim_{x\to 4^+} f(x)=9$

c. $\lim_{x\to -3^-} f(x)=3-(-3)$
$=6$

d. $\lim_{x\to 4^-} f(x)=2(4)+12$
$=20$

e. $\lim_{x\to -3} f(x)=6$

f. $\lim_{x\to 4} f(x)=$ DNE

g. $\lim_{x\to 7^+} f(x)=9$

h. $\lim_{x\to -5} f(x)=3-(-5)$
$=8$

i. $\lim_{x\to 2} f(x)=2(2)+12$
$=16$

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. $\lim_{x\to c} f(x)=L \Leftrightarrow \lim_{x\to c^+} f(x)=L$ and $\lim_{x\to c^-} f(x)=L$

3. Find the limits below using the graph:

a. $f(1) =$ 2

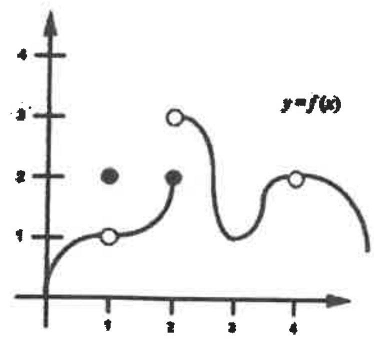
b. $\lim_{x \to 1} f(x) =$ 1

c. $\lim_{x \to 2^+} f(x) =$ 3

d. $\lim_{x \to 2^-} f(x) =$ 2

e. $\lim_{x \to 4} f(x) =$ 2

f. $f(4) =$ DNE



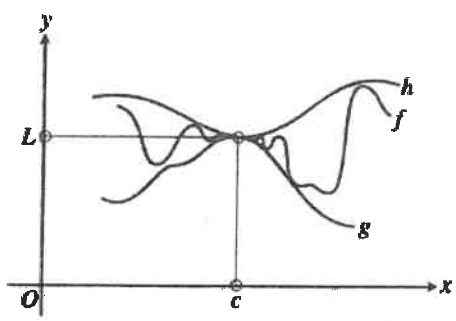
Sandwich Theorem

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} f(x) = L$$



17. Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

sine bounded between -1 and 1

$-1 \leq \sin 1/x \leq 1$

$-x^2 \leq x^2 \sin 1/x \leq x^2$

$\lim_{x \to 0} -x^2 \leq \lim_{x \to 0} x^2 \sin 1/x \leq \lim_{x \to 0} x^2$

$0 \leq \lim_{x \to 0} x^2 \sin 1/x \leq 0$

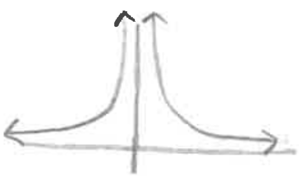
$\lim_{x \to 0} x^2 \sin 1/x = 0$

Horizontal Asymptote:

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either
$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

1. Find the limit and confirm your answer using the Sandwich Theorem

   a. $\lim\limits_{x\to-\infty} \frac{1-\cos x}{x^2} = \lim\limits_{x\to-\infty} \left(\frac{1}{x^2} - \frac{\cos x}{x^2}\right)$



$1/x^2$

$= \lim\limits_{x\to-\infty} \frac{1}{x^2} - \lim\limits_{x\to-\infty} \frac{\cos x}{x^2}$

sandwich thm

$= 0 - \left[-1 \le \cos x \le 1\right]$

$= 0 - \left[-1/x^2 \le \frac{\cos x}{x^2} \le 1/x^2\right]$

$= 0 - \left[\lim\limits_{x\to-\infty} -1/x^2 \le \lim\limits_{x\to-\infty} \frac{\cos x}{x^2} \le \lim\limits_{x\to-\infty} 1/x^2\right]$

$= 0 - \left[0 \le \lim\limits_{x\to-\infty} \frac{\cos x}{x^2} \le 0\right]$

$= 0 - 0$

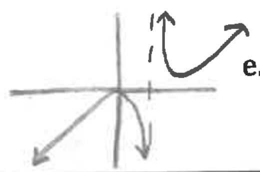
$\boxed{= 0}$

Check the powers of the numerator and denominator.
1) If the denominator (bottom) is a bigger power the limit = 0.
2) If the numerator (top) is a bigger power the limit = ∞ or $-\infty$. slant asym.
3) If powers are the same the limit = $\dfrac{\text{coefficient of the highest power of numerator}}{\text{coefficient of the highest power of denominator}}$

   b. $\lim\limits_{x\to\infty} \frac{2x^2+3}{5x^2-7} = 2/5$

   c. $\lim\limits_{x\to\infty} \frac{6x}{3x^2+1} = 0$

   d. $\lim\limits_{x\to-\infty} \frac{4x^2}{8-3x^2} = -4/3$



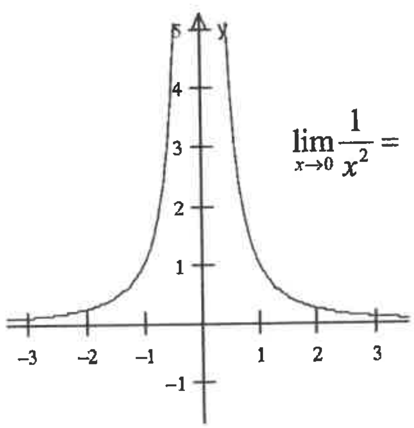
   e. $\lim\limits_{x\to-\infty} \frac{x^2}{3x-1} = -\infty$

   $\approx \frac{x^2}{3x} = \frac{1}{3}x$ acts like linear func. out to ∞

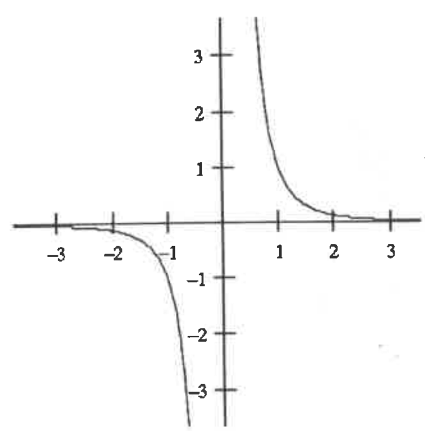
Vertical Asymptote:

The line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either
$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

Unbounded Behavior

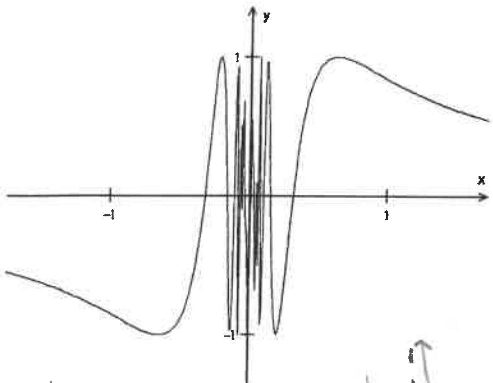


$\lim\limits_{x\to0} \frac{1}{x^2} = \infty$

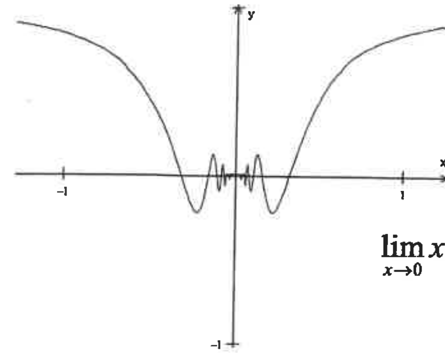


$\lim\limits_{x\to0} \frac{1}{x^3} = \text{DNE}$

3) Oscillating Behavior

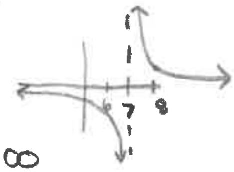


*Exception to Oscillating Behavior



$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

choose values on side of asymptote, if:
1) pos → ∞
2) neg → $-\infty$



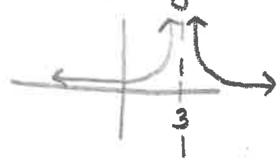
1. $\lim\limits_{x\to 7^+} \frac{1}{x-7} = \infty$

2. $\lim\limits_{x\to 7^-} \frac{1}{x-7} = -\infty$

3. $\lim\limits_{x\to 7} \frac{1}{x-7} =$ DNE

4. $\lim\limits_{x\to 3} \frac{2}{(x-3)^2} = \infty$

* 2nd degree



5. $\lim\limits_{x\to 8} \frac{-6}{(x-8)^2} = -\infty$

6. $\lim\limits_{x\to \frac{\pi}{2}^+} \tan x = -\infty$

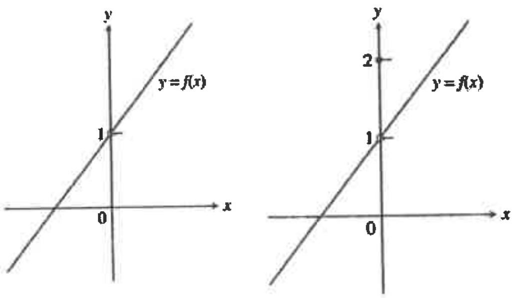
neg S A / T C $\pi/2$ or

Continuity
A function is continuous if:

1. $\lim\limits_{x\to c} f(x)$ exists — * left & right limits are equal
2. $f(c)$ exists
3. $\lim\limits_{x\to c} f(x) = f(c)$

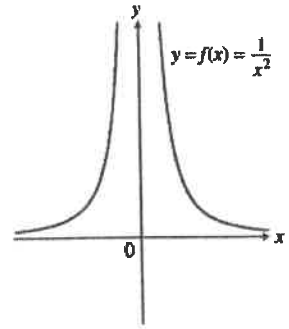
If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f.

1. Removable Discontinuity:

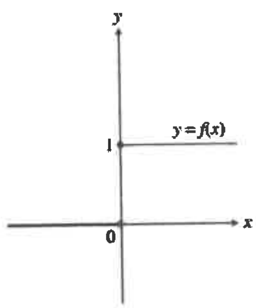


limit exists

3. Infinite Discontinuity:

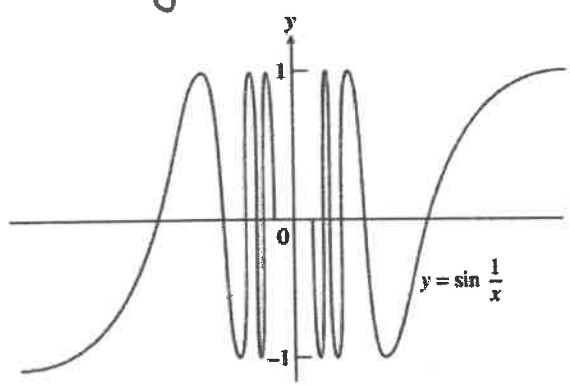


2. Jump Discontinuity:



one sided limits exist

4. Oscillating Discontinuity:

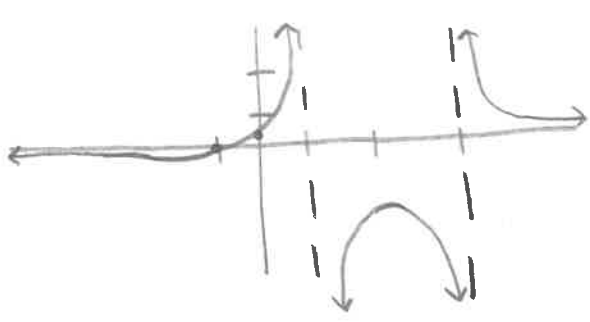


2. Find the points of continuity and the points of discontinuity of the function. Identify each type of discontinuity.

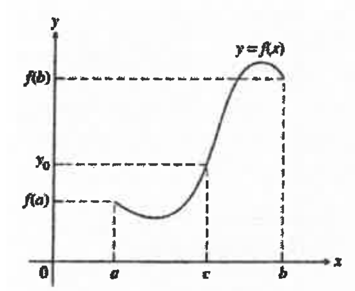
a. $y = \frac{x+1}{x^2-4x+3}$

$= \frac{x+1}{(x-3)(x-1)}$

infinite disc @ $x=1$ and $x=3$



Intermediate Value Theorem
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



2. Find each point of discontinuity for the function below. Then if there are any, determine if the discontinuities are removable.

$$f(x)=\begin{cases}-2x, & x\le 2\\ x^2-4x+1, & x>2\end{cases},\quad x=2$$

$\lim_{x\to 2^-} f(x) = -4$ $\qquad$ $\lim_{x\to 2^+} f(x) = 2^2-4(2)+1$ $\qquad$ $f(2) = -4$

$= -3$

Discontinuous @ $x=2$

not removable, is a jump discontinuity

3. What value should be assigned to k to make f a continuous function?

$$f(x)=\begin{cases}\dfrac{x^2+2x-15}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$$

$\dfrac{x^2+2x-15}{x-3} = \dfrac{(x+5)(x-3)}{x-3}$

$= x+5,\ x \neq 3$

removable disc.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \cancel{\dfrac{3^2+2(3)-15}{3-3}}$

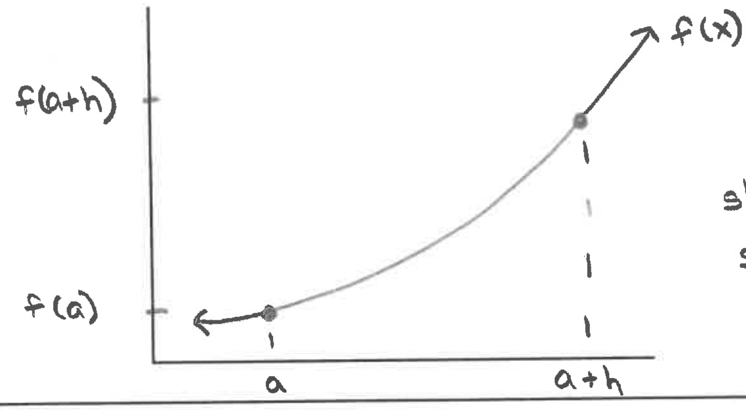
$= 3+5$

$= 8$

$\lim_{x\to 3} f(x) = f(3)$

$\boxed{8 = k}$

Tangent Line



slope of secant = ______

Slope of the tangent Line to a Curve at a Point
The slope of the tangent line to the graph of a function $y = f(x)$ at $(a,\ f(a))$ is given by:

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Provided that this limit exists. This limit describes:

- The slope of the graph of f at $(a,\ f(a))$
- The instantaneous rate of change of f with respect to x at a

1. Find the derivative of the following:
   a. $\lim\limits_{h \to 0} \frac{(x+h)^3 - x^3}{h}$

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

Average Rate of Change:
The average rate of change of f from $x = a$ to $x = a + h$ is given by the difference quotient:

$$\frac{f(a+h)-f(a)}{h}$$

Instantaneous Rate of Change:
The instantaneous rate of change of f with respect to x at a is the derivative of f at a

$$\lim_{x \to a} \frac{f(a+h)-f(a)}{h}$$